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Functionals of Lévy processes and their applications

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Contents

Abstract

Let $\xi = (\xi_s)_{s\geq 0}$ be a one-dimensional Lévy process. For any $t \in (0, \infty]$, functionals of the type

$$I_\xi(t) := \int_0^t e^{-\xi_s}\mathrm{d}s$$

are often called exponential functionals of Lévy processes. These random variables appear in the study of different topics such as branching processes in random environments, self-similar Markov processes, financial mathematics, and many others, as illustrated in Section 7.

Two of the major results in this thesis are the asymptotics of the following related quantities.

1. In the case where ξ is an increasing process, under a mild condition, the density of $I_\xi(\mathbf{e}_q)$ and its respective derivatives, where $q > 0$ and $\mathbf{e}_q \sim Exp(q)$ is independent of ξ.
2. In the case where $\mathbb{E}[\xi_1] < 0$ and $\mathbb{P}(\xi_1 > t) \approx 1/t^\alpha$ for $\alpha > 1$, the distributional limit of a scaled version of $I_\xi(t)$.

To obtain them, we use a variety of analytic and probabilistic arguments. For the first problem, this includes a saddle point-type method to the Mellin transform of the quantity in question. For the second one, this involves Tauberian and regular variation theory, Laplace and Mellin inversion, potential theory, and complex and asymptotic analysis. However, in both cases, it would be impossible to argue purely analytically and probabilistic thinking is combined with the techniques listed above throughout the proofs, which enables us to draw conclusions using this diverse apparatus.

A key object throughout the thesis will be the Bernstein-gamma functions, defined as the solutions of the equation

$$f(z+1) = \phi(z)f(z), \quad f(1) = 1,$$

where ϕ is a Bernstein function. This class of functions is of utmost importance, as the Mellin transform of the exponential functional solves the equation above, so a good understanding of its solution basically characterises fully the law of the exponential functional. That is why we are driven to obtain many new properties and interpretations of these functions.

Note that when $\phi(z) = z$, the solution of the above equation is the classical gamma function, motivating the given name. Furthermore, we anticipate that our findings on Bernstein-gamma functions will prove valuable in various other contexts, just like the classical gamma function has been.

Chapter I

Introduction

Lévy processes are a natural generalisation in continuous time of classical random walks, that is, processes $(S_n)_{n\in\mathbb{N}}$ with $S_n := X_1 + \cdots + X_n$ for independent and identically distributed (iid) $(X_i)_{i\in\mathbb{N}}$. The most prominent example of them is the famous Brownian motion hinting at their usefulness in modelling, for example, diffusion, fragmentation or financial markets, but also mathematical structures such as random graphs or maps. More details and examples in Section 2.

Let $\xi = (\xi_s)_{s\geq 0}$ be a one-dimensional Lévy process. For any $t \in (0,\infty)$, the random variable

$$I_\xi(t) := \int_0^t e^{-\xi_s}\mathrm{d}s \tag{0.1}$$

is called the exponential functional of Lévy processes on deterministic horizon. Except for a fixed t, what is also of interest are exponential functionals when $t = \infty$ or $t = \mathbf{e}_q$ for some $q > 0$ and $\mathbf{e}_q \sim Exp(q)$ independent of ξ:

$$I_\xi(\mathbf{e}_q) = \int_0^{\mathbf{e}_q} e^{-\xi_s}\mathrm{d}s, \quad I_\xi := I_\xi(\infty) := \int_0^\infty e^{-\xi_s}\mathrm{d}s. \tag{0.2}$$

Some of the first studies on these objects include [Duf89, Duf90, Urb92, Urb95] and since then they have emerged in different areas of probability theory such as option pricing, self-similar fragmentations, self-similar Markov processes, and Lévy processes in random environments. A classical survey of the area is the work of Bertoin and Yor [BY05], however, in the last 20 years, much progress has been made, and we introduce some of the main results and methods in Chapter II.

The thesis presents new results on the following asymptotics:

1. **Densities of exponential functionals of subordinators.**

   It is known that when ξ is a non-decreasing Lévy process, or also called a *subordinator*, the random variable $I_\xi(\mathbf{e}_q)$ has in almost all cases an infinitely differentiable density with respect to the Lebesgue measure on $[0,\infty]$, which we denote by $f_{I_\xi(\mathbf{e}_q)}$. If the drift of the process, see (2.4) for a definition of a drift, is positive, this random variable has a finite support, and if it is zero, it is supported on $(0,\infty)$. In the latter case, we obtain in Theorem 17.1 that under the mild condition of *positive increase*, see (14.3), the asymptotics, for $n \geq 0$ and $x \to \infty$,

$$f^{(n)}_{I_\xi(\mathbf{e}_q)}(x) \overset{\infty}{\approx} \frac{C_\phi \varphi_*^n(x)\sqrt{\varphi_*'(x)}}{x^n} e^{-\int_{\phi_*(1)}^x \frac{\varphi_*(y)}{y}\mathrm{d}y},$$

with explicit C_ϕ, whose sign depends on n, ϕ the Laplace exponent of the process, defined, for $\lambda > 0$, by

$$\mathbb{E}\left[e^{-\lambda\xi_1}\right] =: e^{-\phi(\lambda)},$$

and

$$\phi_*(z) := \frac{z}{\phi(z)}, \quad \varphi_* := \phi_*^{-1}.$$

This and other related results are presented in Chapter IV and are based on the joint work with Mladen Savov [MS23].

2. **Exponential functionals on deterministic horizon of Lévy processes which drift to minus infinity.**

   Let ξ be a Lévy process, which drifts to minus infinity. In this case, the exponential functional $I_\xi(t)$ tends to infinity as $t \to \infty$. Under the assumptions of a finite mean and regularly varying right tail of the process, that is,

   $$\mathbb{E}[\xi_1] \in (-\infty, 0), \quad \text{and} \quad \mathbb{P}(\xi_1 > t) \overset{\infty}{\sim} \frac{\ell(t)}{t^\alpha} \quad \text{for some } \alpha > 1 \text{ and } \ell \text{ slowly varying},$$

   see Appendix B for the definition of slow variation, in Theorem 23.1, we obtain that properly rescaled, the law of I_ξ has a weak limit: for any $a \in (0,1)$,

   $$\frac{\mathbb{P}(I_\xi(t) \in \mathrm{d}y)}{y^a \mathbb{P}(\xi_1 > t)} \xrightarrow[t\to\infty]{w} \nu_a(\mathrm{d}y),$$

   where ν_a is a finite measure, supported on $(0, \infty)$.

   We present the joint work with Mladen Savov on this problem in Section V.

An important tool in both Chapter IV and V are the Bernstein-gamma functions, introduced by Patie and Savov in [PS18], and which we present in Chapter III. In order to use them in the context of the two considered scenarios, we have obtained a lot of new information about them, some of which concern, for example, their Stirling-type asymptotics, see Lemma 19.1, Corollary 19.5, and the proof of Theorem 17.1. Furthermore, using the two-dimensional generalisation of the Bernstein-gamma functions $W_\phi(q, z)$, we determine conditions on which their q-derivatives at zero are finite, see Theorem 24.3, and link them with the harmonic potential of an associated Lévy process, see Lemma 24.1.

1 Mathematical framework and notation

The main objects we will work with are one-dimensional real-valued random processes, so standardly we will assume working on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = D([0,\infty))$ the space of right-continuous functions with left limits, usually abbreviated from French as *càdlàg*. This set can be endowed with the Skorohod topology, and we choose $\mathcal{F}$ to be the induced Borel σ-algebra. We note with $\mathbb{E}$ and Var, respectively, the expectation and variance under $\mathbb{P}$.

We will consider random processes as random maps X on the constructed $(\Omega, \mathcal{F}, \mathbb{P})$. As the trajectories ω are functions $\omega : [0, \infty) \to \mathbb{R}$, we note $X_t(\omega) := X(\omega(t))$. For a sequence of stochastic processes $\{X^{(\lambda)}, \lambda \geq 0\}$, we note the convergence in law on $(\Omega, \mathcal{F}, \mathbb{P})$, see Kallenberg [Kal21, Chapter 23], as

$$X^{(\lambda)} \xRightarrow[\lambda\to\infty]{d} X,$$

and the weaker notion of finite-dimensional convergence

$$X^{(\lambda)} \xrightarrow[\lambda\to\infty]{d} X,$$

defined, with $n \geq 1$ and $t_1 \leq \cdots \leq t_n$, by

$$\left(X_{t_1}^{(\lambda)}, X_{t_2}^{(\lambda)}, \ldots, X_{t_n}^{(\lambda)}\right) \xrightarrow[\lambda\to\infty]{d} (X_{t_1}, X_{t_2}, \ldots, X_{t_n}).$$

For set of numbers, we will note $\mathbb{R}_+ := [0, \infty)$; for $A \subset \mathbb{R}$, $\mathbb{C}_A := \{z \in \mathbb{C} : \mathtt{Re}(z) \in A\}$. Variables noted m, n, k are assumed to be integers unless stated otherwise, similarly z is assumed to be a complex number, and all other variables are assumed to be real if nothing is specified. We denote by $\times$ the product of independent random variables.

For $f, g : \mathbb{R} \to \mathbb{R}$ and $a \in [-\infty, \infty]$, we use the following notation for asymptotic relations:

- $f(x) = \mathrm{o}(g(x))$ as $x \to a$, if $\lim_{x\to a} f(x)/g(x) = 0$;
- $f(x) = \mathrm{O}(g(x))$ as $x \to a$, if $\limsup_{x\to a} |f(x)/g(x)| < \infty$;
- $f \sim g$ as $x \to a$, or also noted as $f \overset{a}{\sim} g$, if $\lim_{x\to a} f(x)/g(x) = 1$;
- $f \propto g$ as $x \to a$ or also noted as $f \overset{a}{\propto} g$, if $f \overset{a}{\sim} cg$ for some $c \in (0, \infty)$.

We will work mainly with asymptotics on infinity, so a from the above definitions will be assumed to be equal to ∞ unless stated otherwise.

We use italics to introduce terms throughout the text.

For general arrangement and style, we tried our best to follow the advice of the 2017 AMS Style Guide.

2 Lévy processes

We recall the definition of a Lévy process: we call $\xi := (\xi_t)_{t\geq 0}$ a *Lévy process* if

1. $\xi_0 = 0$ a.s.;
2. ξ has independent increments;
3. ξ has stationary increments;
4. ξ has càdlàg trajectories.

The law of ξ is described analytically by the Lévy-Khintchine formula: for each $t \geq 0$ and at least for $\mathtt{Re}(z) = 0$,

$$\mathbb{E}\left[e^{z\xi_t}\right] =: e^{t\Psi(z)}, \tag{2.2}$$

with

$$\Psi(z) = \gamma z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \big(e^{zx} - 1 - zx\mathbb{1}_{\{|x|\leq 1\}}\big)\Pi(\mathrm{d}x).$$

for $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and Π a measure supported on $\mathbb{R} \setminus \{0\}$ with

$$\int_{\mathbb{R}} \min\{1, x^2\}\Pi(\mathrm{d}x) < \infty.$$

The function Ψ is called the *characteristic exponent*, or also the *Lévy exponent*, of ξ. Also, the quantities γ, σ and Π are referred usually, respectively, as the *linear coefficient*, *Brownian component*, and the *Lèvy measure* of the process. It is immediate that ξ can be constructed as the sum of 4 independent components,

$$\xi_t = \gamma t + \sigma B_t + \xi_t^{(1)} + \xi_t^{(2)},$$

with γt a deterministic linear term, σB_t a scaled standard Brownian motion, $\xi_t^{(1)}$ a CPP with "large" jumps whose absolute values are larger than 1, and $\xi_t^{(2)}$ a martingale obtained as a compensated CPP with "small" jumps with values in $[-1, 1]$.

Let $\mathbf{e}_q$ be an exponentially distributed random variable with parameter $q \geq 0$, which is independent of ξ. In the case $q = 0$, we define $\mathbf{e}_q$ as ∞. We define the killed process $\widetilde{\xi}$ to be equal to

$$\widetilde{\xi}_t := \begin{cases} \xi_t, & \text{for } t < \mathbf{e}_q; \\ \infty, & \text{for } t \geq \mathbf{e}_q. \end{cases}$$

The exact value ∞ is not important, as its role is to describe the so-called *cemetery state*. The Lévy-Khintchine of this new process is thus

$$\Psi(z) = -q + \gamma z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \big(e^{zx} - 1 - zx\mathbb{1}_{\{|x|\leq 1\}}\big)\Pi(\mathrm{d}x). \tag{2.3}$$

We call the unkilled process ξ *conservative* which corresponds also to a killed process with rate 0. We note that from now on, in most cases, we will be working with killed processes.

2.1 Some classes of Lévy processes

Subordinators. The class of processes which are the main object of interest in Chapter IV is the one encompassing non-decreasing Lévy processes. Such processes are named *subordinators* following the work of Bochner [Boc55].

What is of vital importance when working with subordinators is that, since they are non-negative processes, we can work with their Laplace exponent. If ξ is a subordinator, which is potentially killed at rate q, we can define, for each $\lambda \geq 0$,

$$\phi(\lambda) := \ln\big(\mathbb{E}\big[e^{-\lambda\xi_1}\big]\big) = q + \mathtt{d}\lambda + \int_{(0,\infty)} \big(1 - e^{-\lambda y}\big)\mu(\mathrm{d}y), \tag{2.4}$$

with the Lévy measure μ, supported on $(0, \infty)$, satisfying

$$\int_{(0,\infty)} \min\{1, y\}\mu(\mathrm{d}y) < \infty.$$

The term $\mathtt{d}$ is called the *drift* of the process because, for $t < \mathbf{e}_q$, we can write

$$\xi_t = \mathtt{d}t + \sum_{0\le s\le t} \Delta_s,$$

with Δ a Poisson point process with characteristic measure μ.

The class of all such functions ϕ is also known in analysis under the name *Bernstein functions.*

We note also that by analytic extension, we can extend the function ϕ to the entire positive half-plane, that is, for $\mathtt{Re}(z) \ge 0$, we define

$$\phi(z) := \log_0\big(\mathbb{E}\big[e^{-z\xi_1}\big]\big) = q + \mathtt{d}z + \int_{(0,\infty)} \big(1 - e^{-zy}\big)\mu(\mathrm{d}y), \tag{2.5}$$

with $\log_0$ being the principal branch of the complex logarithm, and the function is analytic on $\mathtt{Re}(z) > 0$ and continuous on $\mathtt{Re}(z) \ge 0$.

We also introduce the two-dimensional subordinators $\xi := (\xi^{(1)}, \xi^{(2)})$ being $[0,\infty)^2$-valued processes with stationary and independent increments, and each component having càdlàg trajectories. We define their Laplace exponent, for $\alpha, \beta \ge 0$,

$$\kappa(\alpha,\beta) := \ln\Big(\mathbb{E}\Big[e^{-\alpha\xi_1^{(1)} - \beta\xi_1^{(2)}}\Big]\Big) = q + \mathtt{d}_1\alpha + \mathtt{d}_2\beta + \int_{(0,\infty)^2} \big(1 - e^{-\alpha x - \beta y}\big)\mu(\mathrm{d}x,\mathrm{d}y), \tag{2.6}$$

with $q \ge 0$ the killing rate, $\mathtt{d}_1, \mathtt{d}_2 \ge 0$ the drift terms, and Lévy measure μ satisfying

$$\int_{(0,\infty)^2} \min\Big\{1, \sqrt{x^2+y^2}\Big\}\mu(\mathrm{d}x,\mathrm{d}y) < \infty.$$

As in the univariate case, we can extend κ analytically to $\mathbb{C}_{(0,\infty)} \times \mathbb{C}_{(0,\infty)}$ and continuously on $\mathbb{C}_{[0,\infty)} \times \mathbb{C}_{[0,\infty)}$.

In this section of the thesis are also presented the classes of *convolution equivalent* and *subexponential* Lévy processes.

3 The ladder process and the Wiener-Hopf factorisation

We present fundamental results regarding the so-called ladder process. Let us first define the running supremum process:

$$S_t := \sup_{s\le t} \xi_t,$$

and $L := (L_t)_{t\ge 0}$ be a *local time at the maximum.* Note that if L is such a process, then cL for $c > 0$ would also be a local time at the maximum. To obtain the times at which ξ attains a maximum, let us introduce the inverse local time process, for $L_\infty := \lim_{t\to\infty} L_t$, define

$$L_t^{-1} := \begin{cases} \inf\{t \ge 0 : L_s > t\}, & \text{if } t < L_\infty; \\ \infty, & \text{otherwise,} \end{cases}$$

which is also called the *ascending ladder time process.* Next, we introduce the *ascending ladder height process* as

$$H_t := \begin{cases} S_{L_t^{-1}} = X_{L_t^{-1}}, & \text{if } t < L_\infty; \\ \infty, & \text{otherwise.} \end{cases}$$

It turns out that L_∞ is exponentially distributed. If we call its parameter q, we obtain that *the ascending ladder process* (L^{-1}, H) is a bivariate subordinator killed at rate q. Similarly, we define the *descending ladder process* $(\widehat{L}^{-1}, \widehat{H})$ as the ascending ones of $-\xi$. Let us define the Laplace exponents of the two ladder processes to be, for $\mathtt{Re}(\alpha), \mathtt{Re}(\beta) \geq 0$,

$$\kappa_+(\alpha, \beta) := \ln\Big(\mathbb{E}\Big[e^{-\alpha L_1^{-1} - \beta H_1}\Big]\Big), \quad \text{and} \quad \kappa_-(\alpha, \beta) := \ln\Big(\mathbb{E}\Big[e^{-\alpha \widehat{L}_1^{-1} - \beta \widehat{H}_1}\Big]\Big).$$

These two functions have a very convenient representation in terms of the law of ξ due to a result of Fristedt [Fri74] who established that

$$\kappa_\pm(q, z) = c_\pm \exp\left(\int_0^\infty \int_{[0,\infty)} \left(\frac{e^{-t} - e^{-qt-zx}}{t}\right) \mathbb{P}(\pm\xi_t \in \mathrm{d}x)\mathrm{d}t\right), \tag{3.2}$$

where c_+ and c_- are some positive constants which depend on the choien local time. This in particular means that using a particular version of L, we can fix either one of $c_+c_- = 1$, $c_+ = 1$, or $c_- = 1$. It turns out that in the case where ξ is not a CPP,

$$\Psi(z) - q = -\kappa_+(q, -z)\kappa_-(q, z),$$

which is called *Wiener-Hopf factorisation* of Ψ. It represents Ψ as the product of two functions of z such that the first is analytic on $\mathtt{Re}(z) < 0$ and the second on $\mathtt{Re}(z) > 0$. Such types of factorisation are useful when solving various types of partial differential equations were introduced by Wiener and Hopf about 1931 in [WH31] while solving a special type of integral equation.

To account for a potential atom at 0 of the law of ξ, let us introduce the function

$$h(q) := \exp\left(-\int_0^\infty \left(\frac{e^{-t} - e^{-qt}}{t}\right) \mathbb{P}(\xi_t = 0)\mathrm{d}t\right).$$

If we define, for $q \geq 0$ and $\mathtt{Re}(z) \geq 0$,

$$\phi_+(q, z) = \exp\left(\int_0^\infty \int_{[0,\infty)} \left(\frac{e^{-t} - e^{-qt-zx}}{t}\right) \mathbb{P}(\xi_t \in \mathrm{d}x)\mathrm{d}t\right),$$
$$\phi_-(q, z) = \exp\left(\int_0^\infty \int_{(0,\infty)} \left(\frac{e^{-t} - e^{-qt-zx}}{t}\right) \mathbb{P}(\xi_t \in -\mathrm{d}x)\mathrm{d}t\right),$$

and fix the local time L such that $c_+c_- = 1$, we obtain the Wiener-Hopf factorisation, for $z \in i\mathbb{R}$,

$$\Psi(z) - q = -\phi_+(q, -z)\phi_-(q, z) = -h(q)\kappa_+(q, -z)\kappa_-(q, z). \tag{3.3}$$

Chapter II

Exponential functionals of Lévy processes

We recall that we defined the exponential functionals of Lévy processes in (0.1) and (0.2) as

$$I_\xi(t) := \int_0^t e^{-\xi_s}\mathrm{d}s, \quad I_\xi(\mathbf{e}_q) = \int_0^{\mathbf{e}_q} e^{-\xi_s}\mathrm{d}s, \quad \text{and} \quad I_\xi := I_\xi(\infty) := \int_0^\infty e^{-\xi_s}\mathrm{d}s. \tag{3.1}$$

4 The exponential functional on infinite horizon, $I_\xi := I_\xi(\infty)$.

We have the following result.

Theorem 4.1 ([BY05, Theorem 1]). *The following assertions are equivalent:*

(i) $I_\xi < \infty$ *a.s.;*

(ii) $\mathbb{P}(I_\xi < \infty) > 0$*;*

(iii) $\lim_{t\to\infty} \xi_t = \infty$ *a.s.;*

(iv) $\lim_{t\to\infty} \xi_t/t > 0$ *a.s.;*

(v) $\int_1^\infty \mathbb{P}(\xi_t \leq 0)\mathrm{d}t/t < \infty$.

4.1 Link with generalised Ornstein-Uhlenbeck processes.

The first major progress in the area of exponential functionals was the work of Carmona, Petit, and Yor, [CPY97] which establish:

1. in [CPY97, Proposition 2.1] that if ξ is such that $\mathbb{E}[\xi_1] > 0, \mathbb{E}[|\xi_1|] < \infty$, and

$$\xi_t = ct + \sigma B_t + \tau_t^+ - \tau_t^-,$$

where $c \in \mathbb{R}$, B is a Brownian motion, $\tau^\pm$ are subordinators, and the three processes are mutually independent, then I_ξ has infinitely divisible density which solves a integro-differential equation, see [CPY97, (1.1)].

The conclusions of [CPY97] are drawn by characterising the law of I_ξ as the stationary measure of a generalised Ornstein-Uhlenbeck process. This family of processes is constructed as follows: for a Lévy process ξ, the *generalised Ornstein-Uhlenbeck process* U_ξ is defined, for $t \geq 0$ and $x \geq 0$, by

$$U_t^\xi(x) = xe^{-\xi_t} + e^{-\xi_t}\int_0^t e^{\xi_s}\mathrm{d}s \stackrel{d}{=} xe^{\xi_t} + \int_0^t e^{-\xi_s}\mathrm{d}s.$$

For basic information on these processes, see Appendix 1 of [CPY97]. Therefore, when ξ drifts to infnity, and thus by Theorem 4.1, I_ξ is finite, we see that

$$U_\infty^\xi(x) \stackrel{d}{=} I_\xi.$$

This approach is generalised in [CPY01, Theorem 3.1], linking the invariant distribution of a two-dimensional generalised Ornstein-Uhlenbeck processes, i.e. for a two-dimensional Lévy process (ξ, η),

$$U_t^{\xi,\eta}(x) = xe^{-\xi_t} + e^{-\xi_t}\int_0^t e^{\xi_s}\mathrm{d}s \stackrel{d}{=} xe^{\xi_t} + \int_0^t e^{-\xi_s}\mathrm{d}\eta_s$$

with the law of $\int_0^\infty e^{-\xi_t}\mathrm{d}\eta_s$.

The link with one-dimensional generalised Ornstein-Uhlenbeck processes was also used in [PPS12, Theorem 1.2] to obtain that, under mild conditions if ξ drifts to ∞, I_ξ can be factorised as

$$I_\xi \stackrel{d}{=} I_H \times I_{Y_{\widehat{H}}}, \tag{4.3}$$

where H is the ascending ladder height process and $Y_{\widehat{H}}$ is an independent process of H, which has only positive jumps, constructed by the descending ladder height process $\widehat{H}$. A generalisation of the last result that is true for all ξ,

$$I_\xi \stackrel{d}{=} I_H \times X_{\widehat{H}},$$

with $X_{\widehat{H}}$ a positive random variable is obtained in [PS18, Theorem 2.22], and we consider this in more detail in Section III.

2. in [CPY97, Proposition 3.1] the equation, for $z \in [0, \infty)$ such that $\Psi(-z) < 0$

$$\mathbb{E}\left[I_\xi^z\right] = \frac{-z}{\Psi(-z)}\mathbb{E}\left[I_\xi^{z-1}\right].$$

This equation was further analysed by Maulik and Zwart in [MZ06] and understood in depth for complex z in [PS18] motivating the definition of Bernstein-gamma functions.

4.2 Connection with self-similar Markov processes. The Lamperti transform.

We have mentioned that the explicit law of the exponential functional is usually not available as it is in the case where the underlying process ξ is a Brownian motion with positive drift, (4.1). However, it turns out that in some cases it can be linked with the distribution at time 1 of a specific class of random processes called positive self-similar.

We call a $[0,\infty)$-valued strong Markov process X *positive self-similar Markov process with index of self-similarity* $1/\alpha$ (or also $1/\alpha$*-self-similar*) if there exists $\alpha > 0$ such that, for any $x > 0$ and $c > 0$,

$$\text{the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ under } P_x \text{ is } P_{cx}, \tag{4.4}$$

where P_x is the law of X starting at x. If it is possible to define P_0, we obtain that under P_0,

$$(X_{kt}, t \geq 0) \stackrel{d}{=} (k^\alpha X_t, t \geq 0).$$

A fundamental result of Lamperti [Lam62] shows the universality of self-similar processes:

Theorem 4.2 ([Lam62, Theorem 2] in the form of [EM00, Theorem 3.1])**.** *Suppose X is a stochastic process which is a.s. continuous at $t = 0$ and the law of X_t is non-degenerate for each $t > 0$. If there exist a stochastic process Y and real numbers $\{a(\lambda), \lambda \geq 0\}$ with $a(\lambda) > 0, \lim_{\lambda\to\infty} a(\lambda) = \infty$ such that*

$$\frac{1}{a(\lambda)} Y_{\lambda t} \xrightarrow[\lambda\to\infty]{d} X_t,$$

then X is self-similar with some index of similarity $\alpha > 0$. Moreover $a(\lambda) = \lambda^\alpha \ell(\lambda)$ for some slowly varying function ℓ (see Appendix A for definition).

We recall that the famous Donsker theorem, e.g. [Kal21, Theorem 22.9], states that if $X_1, X_2, \ldots$ are iid with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$, then

$$\frac{1}{n^{1/2}} \sum_{j=1}^{\lfloor nt \rfloor} X_j \xRightarrow[n\to\infty]{d} B_t, \tag{4.5}$$

and that the Brownian motion is a 1/2-self-similar process. Therefore, Theorem 4.2 characterises general self-similar processes as those which appear in limits similar to (4.5).

In 1972, Lamperti proved in [Lam72, Theorem 4.1] that there is a bijective correspondence between positive α-self-similar Markov processes and Lévy processes for each $\alpha > 0$, see (4.6) and (4.7) in the thesis.

By the scaling property, (4.4) the asymptotics of self-similar processes at infinity are linked with the limiting behaviour of P_x at 0+: it holds that there exists a probability measure P_0, called *entrance law from 0+*, such that, for any $y > 0$,

$$P_y(t^{-1/\alpha} X_t \in \mathrm{d}x) \xrightarrow[t\to\infty]{w} P_0(X_1 \in \mathrm{d}x)$$

if and only if

$$P_{x_0}(X_1 \in \mathrm{d}x) \xrightarrow[x_0\to 0+]{w} P_0(X_1 \in \mathrm{d}x).$$

We have the following result, using the notation P_x and X as above:

Theorem 4.3. *[BY02a, Theorem 1] Suppose ξ is not arithmetic, i.e. does not live on $r\mathbb{Z}$ for some $r > 0$, and $\mathbb{E}[\xi_1] = m > 0$. Then*

(i) the entrance law P_0 exists;

(ii) we have the distributional equality, for each $t \geq 0$ and measurable $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$E_0(f(X_t^\alpha)) = \frac{1}{\alpha m}\mathbb{E}\left[\frac{f(t/I_{\alpha\xi})}{I_{\alpha\xi}}\right].$$

In particular,

$$P_0\left(\frac{1}{X_1^\alpha} \in \mathrm{d}x\right) = \frac{1}{\alpha m}\frac{\mathbb{P}(I_{\alpha\xi} \in \mathrm{d}x)}{x}. \tag{4.8}$$

4.3 Moments of I_ξ

We recall that in order the moments of I_ξ to be finite, by Theorem 4.1, we need at least that ξ drifts to infinity.

Calculating the moments of I_ξ is a subcase of the same problem for $I_\xi(\mathbf{e}_q)$ by setting $q = 0$, and the two problems were usually solved together. However, for a smoother presentation, we will now stick to I_ξ and consider the latter in Section 5.2.

First analysed in some cases and real z by Carmona et al. in [CPY97, Proposition 3.1], the key equation for calculating the moments of I_ξ was identified to be

$$\mathbb{E}\left[I_\xi^z\right] = \frac{-z}{\Psi(-z)}\mathbb{E}\left[I_\xi^{z-1}\right]. \tag{4.10}$$

Of course, whether the quantities in (4.10) are finite depends on the nature of the process ξ. This recurrence equation was understood in depth for respective regions of the complex plane in [PS18] and its solution was given in terms of Bernstein-gamma functions. We note that [PS18, Theorem 2.4] establishes explicit criteria for the finiteness of $\mathbb{E}\left[I_\xi^a\right]$ in terms of the analytic properties of the Wiener-Hopf factors $\phi_\pm$ of Ψ. We provide for completeness classical results which were milestones in the field.

Theorem 4.4. *(i) ([CPY97, Proposition 3.1, Proposition 3.3]) Suppose ξ is a subordinator. Then, for each integer $n \geq 1$,*

$$\mathbb{E}\left[I_\xi^n\right] = \frac{(-1)^n n!}{\Psi(-1)\Psi(-2)\cdots\Psi(-n)}.$$

Moreover, the distribution of I_ξ is determined by the moments above.

(ii) ([BY02b, Proposition 2]) Suppose that ξ drifts to infinity and has exponential moments, that is, for any $\lambda, t \geq 0$,

$$\mathbb{E}\left[e^{\lambda\xi_t}\right] = e^{t\Psi(\lambda)} < \infty.$$

Then, for each integer $n \geq 1$,

$$\mathbb{E}\left[I_\xi^{-n}\right] = \mathbb{E}[\xi_1]\frac{\Psi(1)\Psi(2)\cdots\Psi(n-1)}{(n-1)!}.$$

If, in addition, ξ does not have positive jumps, then the distribution of I_ξ is determined by the moments above.

4.4 Tail asymptotics for I_ξ. Asymptotics of its density

Before we tract the research on the tail asymptotics of I_ξ, we first note that it was established without any restrictions on ξ by Patie and Savov in [PS18, Theorem 2.11] on a logarithmic scale to be

$$\lim_{x\to\infty}\frac{\ln(\mathbb{P}(I_\xi > x))}{\ln(x)} = c \in [-\infty, 0],$$

including the case of killed processes, which means that it holds for the functionals $I_\xi(\mathbf{e}_q)$ too, which we consider in Section 5.

The available results are summarised in the following theorem.

Theorem 4.5. *1. ([Riv05, Lemma 4], [PS18, Theorem 2.11]) If ξ is non-lattice and that there exists a finite $\theta = \mathfrak{u}_- > 0$ such that*

$$\mathbb{E}\left[e^{-\theta\xi_1}\right] = 1, \quad \textit{and} \quad \left|\mathbb{E}\left[\xi_1 e^{-\theta\xi_1}\right]\right| < \infty,$$

then

$$\mathbb{P}(I_\xi > t) \propto t^{-\theta}.$$

2. ([MZ06, Theorem 4.1]) If ξ is not spectrally negative, $\xi \in \mathcal{S}_0$, and $\mathbb{E}[\xi_1] \in (0,\infty)$, then

$$\mathbb{P}(I_\xi > t) \propto \int_{\ln t}^{\infty} \overline{\Pi}_+(s)\mathrm{d}s.$$

3. ([Riv12, Theorem 1], [AR23, Theorem 2.9]) If $\xi \in \mathcal{S}_\alpha$ for some $\alpha > 0$, and $\mathbb{E}\left[e^{\alpha\xi_1}\right] < 1$, then

$$\mathbb{P}(I_\xi > t) \propto \overline{\Pi}_+(\ln t).$$

4. ([Haa24, Theorem 1], [MS23, Theorem 3.1]) If ξ is a driftless subordinator, define the functions ϕ, ϕ_*, and φ_* by

$$e^{-\phi(\lambda)} := \mathbb{E}\left[e^{-\lambda\xi_1}\right], \quad \phi_*(z) := \frac{z}{\phi(z)}, \quad \textit{and} \quad \varphi_* := \phi_*^{-1}.$$

Then if

$$\limsup_{x\to\infty}\frac{x\phi'(x)}{\phi(x)} < 1,$$

it holds that

$$\mathbb{P}(I_\xi > t) \propto \frac{t\sqrt{\varphi_*'(t)}}{\varphi_*(t)} e^{-\int_{\phi_*(1)}^{t}\frac{\varphi_*(y)}{y}\mathrm{d}y}.$$

To state our next result, we need the variable N_Ψ from [PS18, (2.18)]. Its exact value is available in the last reference, but for a lighter presentation, we only note that

$$N_\Psi \in \begin{cases} \{0\} & \text{if } \xi_t = ct \text{ for some } c > 0; \\ (0,\infty) & \text{if } \xi \text{ is a CPP with a positive drift, and } \xi_t \neq ct \text{ for some } c > 0; \\ \{\infty\} & \text{otherwise.} \end{cases} \tag{4.12}$$

Furthermore, we introduce the class of weakly non-lattice processes ξ, introduced in [PS18, (2.29)], as those for which

$$\mathfrak{u}_- \in (-\infty, 0) \quad \text{and} \quad \liminf_{|b|\to\infty} |b|^k |\Psi(\mathfrak{u}_- + ib)| > 0,$$

which is a subclass of the non-lattice one. Let us also define the class C_0^k to be the class of k times differentiable functions with their k derivatives vanishing at infinity. With this in hand, we have the following result.

Theorem 4.6. *1. ([BLM08, Theorem 3.8]) If I_ξ is finite, the distribution function $F_{I_\xi} := t \mapsto \mathbb{P}(I_\xi \le t)$ is differentiable.*

2. ([PS18, Theorem 2.4.3]) The function F_{I_Ψ} belongs to the class $C_0^{\lceil N_\Psi \rceil - 1}$.

3. ([PS18, Theorem 2.11]) If ξ is weakly non-lattice and

$$\left|\mathbb{E}\left[\xi_1^2 e^{\mathfrak{u}_- \xi_1}\right]\right| < \infty,$$

then, for each $n \le \lceil N_\Psi \rceil - 2$,

$$f_{I_\xi}^{(n)}(t) \propto t^{\mathfrak{u}_- - n - 1}.$$

4. ([MS23, Theorem 3.1]) If ξ is a subordinator with positive increase, then, for each $n \ge 0$,

$$f_{I_\xi}^{(n)}(t) \propto \frac{\varphi_*^n(t)\sqrt{\varphi_*'(t)}}{t^n} e^{-\int_{\phi_*(1)}^t \frac{\varphi_*(y)}{y} \mathrm{d}y}.$$

5 The exponential functional at random times $\mathbf{e}_q \sim Exp(q)$, $I_{\xi,q} := I_\xi(\mathbf{e}_q)$

Considering the exponential functional time, extending it to a random exponential time is a natural generalisation of I_ξ. Indeed, when allowing $q = 0$, $I_{\xi,0}$ is precisely I_ξ. The initial motivation for exploring this concept arose from examining the expectation of $I_{\xi,q}$,

$$\mathbb{E}[I_{\xi,q}] = \mathbb{E}\left[\int_0^{\mathbf{e}_q} e^{-\xi_s}\mathrm{d}s\right] = \mathbb{E}\left[\int_0^\infty \int_0^t e^{-\xi_s} q e^{-qt}\mathrm{d}s\mathrm{d}t\right] = q\int_0^\infty e^{-qt}\mathbb{E}[I_\xi(t)]\mathrm{d}t,$$

which is in essence the Laplace transform of $I_\xi(t)$. Analysing the distribution of $I_\xi(t)$ directly is often challenging, and this motivated the result mentioned in (3.4) by Yor, [Yor92a, Theorem 2],

$$2I_{B_\mu}(\mathbf{e}_q) \stackrel{d}{=} \frac{B_{1,\alpha}}{\Gamma_{\beta,1}}$$

for independent $B_{1,\alpha} \sim Beta(1, \alpha)$ and $\Gamma_{\beta,1} \sim Gamma(\beta, 1)$ with $\alpha := (\mu + \sqrt{2\lambda + \mu^2})/2$ and $\beta := \alpha - \mu$. However, it proves challenging to derive other explicit results for the law of $I_{\xi,q}$. Notably, there exist such for a specific type of subordinators by Möhle in [Mö15, Section 2].

We present the results in an order which mimics Section 4 and not chronologically, so that comparisons are immediate.

5.1 Density of $I_{\xi,q}$. Generalisation of the Carmona-Petit-Yor equation and the link with generalised Ornstein-Uhlenbeck proceses

We recall that Bertoin et al. [BLM08, Theorem 3.9] proved that I_ξ has a density, which we note f_{I_ξ}. This was extended to $I_{\xi,q}$ for $q > 0$ by Pardo et al. in [PRvS13, Theorem 2.1], where the CPY equation was also generalised for subordinators in Theorem 2.3 therein. The further extension to $q = 0$ was made by Behme et al., see [BLR21, Remark 5.9].

We have introduced in Section 4.1 the generalised Ornstein-Uhlenbeck processes as

$$U_t^{\xi,\eta}(x) = xe^{-\xi_t} + e^{-\xi_t}\int_0^t e^{\xi_s}\mathrm{d}s \overset{d}{=} xe^{\xi_t} + \int_0^t e^{-\xi_s}\mathrm{d}s,$$

and characterised the exponential functional I_ξ as its stationary distribution. For killed processes, an analogous result was obtained by Behme et al. [BLRR21]. More precisely, first observe that generalised Ornstein-Uhlenbeck processes can be defined as the unique solution of the stochastic differential equation

$$\mathrm{d}X_t = X_{t-}\mathrm{d}U_t + \mathrm{d}t,$$

where the process U is defined implicitly by

$$\mathcal{E}(U)_t = e^{-\xi_t}, \quad \text{with} \quad \mathcal{E}_0 = 1, \mathrm{d}\mathcal{E}(U)_t = \mathcal{E}(U)_t\mathrm{d}U_t. \tag{5.1}$$

Behme et al. [BLRR21] proved that the law of $I_{\xi,q}$ is the unique stationary distribution of the Markov process defined by

$$\mathrm{d}X_t = X_{t-}\mathrm{d}\widetilde{U}_t + \mathrm{d}t,$$

where, with U defined in (5.1) and a Poisson process N_q of parameter $q \geq 0$,

$$\widetilde{U} := U - N_q.$$

As in Section 4.1, the conclusions of this subsection were often drawn for the more general exponential functionals $\int_0^\infty e^{-\xi_t}\mathrm{d}\eta_s$. In particular, for the Carmona-Petit-Yor equation, see [KPS12, BLR21], for tail asymptotics [KPS12], for the support [BLRR21], and [BLR21] for the link with Ornstein-Uhlenbeck processes. For an overview of all these, see the thesis [Rek21].

5.2 Moments of $I_{\xi,q}$.

We have already discussed the case $q = 0$ in Section 4.3, so let us assume now that $q > 0$. In this case, the exponential functional is finite a.s., as noted in [PS18, (2.20)]. Again, an analogue of the recurrence relation (4.10) holds true,

$$\mathbb{E}\big[I_{\xi,q}^z\big] = \frac{-z}{\Psi_q(-z)}\mathbb{E}\big[I_{\xi,q}^{z-1}\big],$$

where

$$\Psi_q(z) := \Psi(z) - q$$

being the characteristic exponent of the killed ξ is vital for understanding the moments of $I_{\xi,q}$, and [PS18, Theorem 2.18] establishes precise conditions for the finiteness of $\mathbb{E}\big[I_{\xi,q}^a\big]$ in terms of the analytic properties of the Wiener-Hopf factors of Ψ.

5.3 Tail asymptotics for $I_{\xi,q}$. Asymptotics for its density

For the case $q = 0$, see Section 4.4. We now assume that $q > 0$ unless stated otherwise.

We have seen that the results on the moments of I_ξ were usually obtained on the more general $I_{\xi,q}$. However, for the tail behaviour, the generalisation came later. The analogue of Theorem 4.5 is the following:

Theorem 5.1. *1. ([Riv05, Lemma 4], [PS18, Theorem 2.11]) If ξ is non-lattice and that there exists $\theta > 0$ such that*

$$\mathbb{E}\left[e^{-\theta\xi_1}\right] = 1, \quad \textit{and} \quad \left|\mathbb{E}\left[\xi_1 e^{-\theta\xi_1}\right]\right| < \infty,$$

then

$$\mathbb{P}(I_{\xi,q} > t) \propto t^{-\theta}.$$

2. (i) ([AR23, Theorem 2.9]) If ξ is not spectrally negative, $\xi \in \mathcal{S}_0$, and $\mathbb{E}[\xi_1] \in (-\infty, 0)$,

or

(ii) $\xi \in \mathcal{S}_\alpha$ for some $\alpha > 0$, and $\mathbb{E}\left[e^{\alpha\xi_1}\right] < 1$,

then

$$\mathbb{P}(I_{\xi,q} > t) \propto \overline{\Pi}_+(\ln t).$$

3. ([MS23, Theorem 3.1]))If ξ is a driftless subordinator, define functions ϕ, ϕ_*, and φ_* by

$$e^{-\phi(\lambda)} := \mathbb{E}\left[e^{-\lambda\xi_1}\right], \quad \phi_*(z) := \frac{z}{\phi(z)}, \quad \textit{and} \quad \varphi_* := \phi_*^{-1}.$$

Then if

$$\limsup_{x\to\infty} \frac{x\phi'(x)}{\phi(x)} < 1,$$

it holds that

$$\mathbb{P}(I_{\xi,q} > t) \propto \frac{t\sqrt{\varphi_*'(t)}}{\varphi_*(t)} e^{-\int_{\phi_*(1)}^{t} \frac{\varphi_*(y)}{y}\mathrm{d}y}.$$

The available asymptotics for the density are practically the same as in the case of non-killed subordinators, see Theorem 4.6.

Theorem 5.2. *1. ([BLM08, Theorem 3.8], [PRvS13, Theorem 2.1], [PS18, Theorem 2.4.3]) The distribution function $F_{I_{\xi,q}} := t \mapsto \mathbb{P}(I_{\xi,q} \leq t)$ belongs to the class $C_0^{\max\{1,\lceil N_\Psi\rceil-1\}}$, with N_Ψ, defined in (4.12).*

2. ([PS18, Theorem 2.11]) If ξ is weakly non-lattice and

$$\left|\mathbb{E}\left[\xi_1^2 e^{\mathrm{u}_-\xi_1}\right]\right| < \infty,$$

then, for each $n \leq \lceil N_\Psi\rceil - 2$,

$$f_{I_\xi}^{(n)}(t) \propto t^{\mathrm{u}_- - n - 1}.$$

3. *([MS23, Theorem 3.1]) If ξ is a subordinator with positive increase, then, for each $n \geq 0$,*

$$f_{I_\xi}^{(n)}(t) \propto \frac{\varphi_*^n(t)\sqrt{\varphi_*'(t)}}{t^n} e^{-\int_{\phi_*(1)}^t \frac{\varphi_*(y)}{y} \mathrm{d}y}.$$

6 The exponential functional on deterministic horizon, $I_\xi(t)$

The exponential functional on deterministic horizon is the hardest object among the three types we are considering, that is, alongside I_ξ and $I_{\xi,q}$. The explicit law of $I_\xi(t)$ is known in a small number of cases, notably for Brownian motion with drift. However, the law, although explicit, is sophisticated; see [AMS01, Yor92a] or the survey [MY05, Section 4].

We start by presenting the general results available for the moments of $I_\xi(t)$.

6.1 Moments of $I_\xi(t)$.

The existence of moments of $I_\xi(t)$ was fully characterised in [PS18, Theorem 2.18] using the characteristics of Ψ, defined in (4.11). However, explicit results are rare. One of them is in the case of a Brownian motion and can be found in [MY05, Section 5]. In a recent work, Palmowski et al. [PSS24, Lemma 2.1] established the convolutional identities between I_ξ and $I_{-\xi}$, for ξ a Lévy process which is not a CPP and $\mathtt{Re}(z) \in (0,1)$,

$$\int_0^t \mathbb{E}\big[I_\xi^{-z}(t-s)\big]\mathbb{E}\big[I_{-\xi}^{z-1}(s)\big]\mathrm{d}s = \frac{\pi}{\sin(\pi z)}.$$

In the symmetric case, that is $\xi \stackrel{d}{=} -\xi$, when ξ is not a CPP, this resulted in the surprising explicit identity that

$$\mathbb{E}\left[\frac{1}{\sqrt{I_\xi(t)}}\right] = \frac{1}{\sqrt{t}}.$$

This was observed by Yor [Yor92a, (1.e)] in the case of a Brownian motion. In addition, Palmowski et al. established in [PSS24, Theorem 2.10] an infinite series expansion for the moments when ξ is a subordinator, extending the work of Barker and Savov [BS21, Theorem 2.14]. For the simple cases of integer moments and a Brownian motion or a CPP, there are also finite sums representations of $\mathbb{E}\big[I_\xi^n\big]$ in Salminen and Vostrikova [SV18, Example 5 and 6]. Such results are convenient as they give a possible numerical method for the computation of the moments. Another possibility is to use the recurrent integral equation system of [SV18, Theorem 1, Theorem 2], see also [PSS24, Remark 2.17].

It should be noted that one of the initial motivations of Yor [Yor92a] behind the introduction of the exponential functional at random exponential times $\mathbf{e}_q$ is that

$$\mathbb{E}\big[I_{\xi,q}^\alpha\big] := \mathbb{E}\big[I_\xi^\alpha(\mathbf{e}_q)\big] = \int_0^\infty q e^{-qt}\mathbb{E}\big[I_\xi^\alpha(t)\big]\mathrm{d}t,$$

which is exactly the Laplace transform of $\mathbb{E}\big[I_\xi^\alpha(t)\big]$. Therefore, using Laplace (or Mellin) inversion techniques, it is possible to recover the moments at a fixed horizon.

6.2 Asymptotic results. Weak limit of the scaled $I_\xi(t)$

In this section, we consider processes for which $I_\xi = \infty$ a.s. By Theorem 4.1, this means that the process ξ either oscillates or drifts to minus infinity. Recent interest in such results stems from their usefulness in analysing branching processes in random environment, which we present in Section 7.1. What is often of interest for applications is the behaviour of expectations

$$\mathbb{E}[F(I_\xi(t))] \quad \text{as} \quad t \to \infty,$$

where $F : \mathbb{R} \to \mathbb{R}$ depends on the specific case. Such results are available in [PPS16, Theorem 1.2], [LX18, Theorem 2.9], [Xu21, Theorem 4.3], and [Xu23] for different classes of functions. Such results are immediate if we can prove that $I_\xi(t)$ has a weak limit in some sense. We first present the two cases where this is possible and, therefore, provide the strongest results.

Theorem 6.1. *1. ([PS18, Theorem 2.20.2]) Let ξ be an oscillating Lévy process, that is $\limsup_{t\to\infty} = -\liminf_{t\to\infty} = \infty$ a.s., which fulfils the Spitzer's condition*

$$\lim_{t\to\infty} \mathbb{P}(\xi_t < 0) = \rho \in [0,1).$$

Then $\kappa_-(q,0) \overset{0}{\sim} q^\rho \ell(q)$ for some $\ell \in SV_0$, and, for any $a \in (0, 1-\mathfrak{a}_+)$,

$$\frac{t^\rho \mathbb{P}(I_\xi(t) \in \mathrm{d}y)}{y^a \ell(1/t)} \xrightarrow[t\to\infty]{w} \widetilde{\nu}_a(\mathrm{d}y),$$

where $\widetilde{\nu}_a$ is a finite measure, supported on $(0,\infty)$, with a distribution function given by

$$\widetilde{\nu}_a((0,x]) = -\frac{1}{2\pi\Gamma(1-\rho)} \int_{\mathtt{Re}(z)=b} \frac{x^{-z}}{z} M_\Psi(0, z+1-a)\mathrm{d}z$$

for all $b \in \mathbb{C}_{(a-1+\mathfrak{a}_+,0)}$. As a consequence, for every function $F : (0,\infty) \to \mathbb{R}$ such that, for some $a \in (0,1)$, $x \mapsto x^a F(x)$ is bounded and continuous,

$$\frac{t^\rho}{\ell(1/t)} \mathbb{E}[F(I_\xi(t))] \xrightarrow[t\to\infty]{} \int_{(0,\infty)} y^a F(y) \widetilde{\nu}_a(\mathrm{d}y) < \infty.$$

2. (Theorem 23.1, Corollary 23.3) Let ξ be a Lévy process with a finite negative mean and

$$\mathbb{P}(\xi_1 > t) \overset{\infty}{\sim} \frac{\ell(t)}{t^\alpha}$$

for some $\alpha > 1$ and $\ell \in SV_\infty$. Then, for any $a \in (0,1)$,

$$\frac{t^\alpha \mathbb{P}(I_\xi(t) \in \mathrm{d}y)}{y^a \ell(t)} \xrightarrow[t\to\infty]{w} \nu_a(\mathrm{d}y),$$

where ν_a is a finite measure, supported on $(0,\infty)$, with a distribution function given by, for any $b \in \mathbb{C}_{(a-1,0)}$,

$$\nu_a((0,x]) = \frac{1}{C_\phi} \int_{\mathtt{Re} z = b} \frac{x^{-z}}{z} M_\Psi(0, z+1-a)\mathrm{d}z \quad \textit{with} \quad C_\phi = -2\pi i \phi_+(0,0)(-\mathbb{E}[\xi_1])^\alpha.$$

As a consequence, for every function $F : (0, \infty) \to \mathbb{R}$ *such that, for some* $a \in (0, 1)$, $x \mapsto x^a F(x)$ *is bounded and continuous,*

$$\frac{t^\alpha}{\ell(t)} \mathbb{E}[F(I_\xi(t))] \xrightarrow[t\to\infty]{} \int_{(0,\infty)} y^a F(y) \nu_a(\mathrm{d}y) < \infty. \tag{6.1}$$

A convenient corollary of Theorem 6.1.1 can be extracted in the case

$$\mathbb{E}[\xi_1] = 0, \quad \text{and} \quad Var(\xi_1) < \infty.$$

Under these conditions, by the central limit theorem, for each step $h > 0$, as $n \to \infty$, $\mathbb{P}(X_{nh} > 0) \to 1/2$. It can then be concluded, for example as in [GS94, Proposition 4.6] that

$$\lim_{t\to\infty} \mathbb{P}(\xi_t < 0) = \rho = 1/2,$$

so the theorem gives

$$\lim_{t\to\infty} \sqrt{t}\mathbb{E}\left[I_\xi^{-a}(t)\right] = C \in (0, \infty), \quad \text{and} \quad \lim_{t\to\infty} \sqrt{t}\mathbb{E}[F(I_\xi(t))] = C_F \in (0, \infty) \tag{6.2}$$

for all F such that $x \mapsto x^a F(x)$ is bounded and continuous.

7 Applications and examples

7.1 Random processes in random environment

One of the first applications of exponential functionals when analysing diffusion processes in random environment is apparent from the works of Brox [Bro86] and Kawazu and Tanaka [KT93], who observed that for a diffusion with a random Brownian potential X, it holds that

$$\mathbb{P}\left(\max_{s\geq 0} X_s > t\right) = \mathbb{E}\left[\frac{I_{B'}}{I_{B'} + I_B(t)}\right],$$

in [KT93], Kawazu and Tanaka obtain the asymptotics of the maximum of this process. This is generalised when the potential V is composed of two Lévy processes in [CPY97, Section 4.1].

Recent interest in processes in a random environment stems from their use in modelling continuous branching processes. In this scenario, exponential functionals appear in the estimation of survival probabilities and explosion rates of the branching process.

7.2 Extinction times of self-similar processes

Another area where exponential functionals are a major tool is the analysis of extinction times of self-similar process. This is in view of the Lamperti transform, which we presented in Section 4.2. We present the standard (α, ν)-fragmentation model in the thesis and discuss related results.

7.3 Financial Mathematics

One of the main motivations behind the initial interest in exponential functionals during the 1990s was the pricing of Asian options and related quantities, such as perpetuities [Duf90] or annuities [GY93].

Consider the following standard model for the price of an asset,

$$A_t := A_0 e^{\xi_t},$$

where ξ is a Lévy process. An Asian option with fixed maturity T is a financial contract with payoff which is determined by the average price of the asset over the period $[0, T]$. Hence, this can be considered as a more safe contract compared to the classical European option, where only the price at time T is relevant. Setting the strike price of the Asian option to K, its payoff is therefore

$$\max\left\{\frac{1}{T}\int_0^T A_s \mathrm{d}s - K, 0\right\} =: \left(\frac{1}{T}\int_0^T A_0 e^{\xi_s} \mathrm{d}s - K\right)_+,$$

using the notation $x_+ := \max\{x, 0\}$. Therefore, the mathematical problem of evaluating the fair price of the option is related to calculating expectations of the type

$$C_\xi(t, a) := \mathbb{E}\big[(I_{-\xi}(t) - a)_+\big]$$

for $a, t > 0$. The explicit evaluation of the last expression turned out to be a difficult task, even in the case of a Brownian motion. See the thesis for a more detailed discussion.

7.4 Other applications

We have already seen in Section 4.2 that the Lamperti transform, see (4.7), is a basic tool in understanding positive self-similar Markov processes, and therefore the relevance of the exponential functionals is natural. Another topic we have already touched upon is the emergence of exponential functionals as stationary distributions of generalised Ornstein-Uhlenbeck processes, see Section 4.1 Exponential functionals appear also as a tool in the studies involving spectral theory of some non-reversible Markov semigroups, see Patie et al. [PSZ19] and Patie and Savov [PS21]. Another interesting occurrence of the exponential functional is the area of non-parametric Bayesian statistics, see [ELP03].

Chapter III

Bernstein-gamma functions

8 Introduction

In this chapter, we present the Bernstein-gamma functions introduced by Patie and Savov in [PS18, Chapter 4].

Let $-\Psi$ be a continuous negative definite function, see Section 11.1 for a definition. Equivalently, this means that Ψ is the characteristic exponent of a Lévy process. It turns out that solving in the space of analytic functions of the functional equation

$$M_\Psi(z+1) = \frac{-z}{\Psi(-z)} M_\Psi(z), \quad M_\Psi(z) = 1 \tag{8.2}$$

can be reduced to solving equations in the class of Mellin transforms of positive random variables of the type

$$W_{\phi_\pm}(z+1) = \phi_\pm(z) W_{\phi_\pm}(z), \quad W_{\phi_\pm}(1) = 1, \tag{8.3}$$

with $\phi_\pm$ members of the class of Bernstein functions. In fact, $\phi_\pm$ are exactly the Wiener-Hopf factors defined by the factorisation, for $z \in i\mathbb{R}$,

$$\Psi(z) = -\phi_+(-z)\phi_-(z),$$

which is the one-dimensional version of (3.3). Having defined the Bernstein-gamma functions $W_{\phi_\pm}$, then we know from [PS18, Theorem 2.1] that a solution of (8.2), at least on some regions of $\mathbb{C}$, is the function

$$M_\Psi = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z),$$

see Section 11 for a precise statement.

The most trivial example of a Bernstein function is the identity $z \mapsto z$. In this case (8.3) becomes the well-known equation

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(1) = 1,$$

which is valid on $\mathbb{C} \setminus \{-1, -2, \dots\}$, whose solution is the famous Euler gamma function. This analogy makes clear the motivation behind the term *Bernstein-gamma functions* for the solutions W_ϕ of (8.3). In the following two sections, we provide more properties of the classical gamma function, which can be generalised to all Bernstein-gamma functions.

9 Representations

9.1 Mellin transform of random variables

We recall that the gamma function can be defined on $\mathtt{Re}(z) > 0$ as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x.$$

Therefore, it is exactly the Mellin transform of an exponentially distributed random variable with parameter 1. An analogue of the latter is true for all Bernstein-gamma functions as by definition, see above (8.3) or Section 11, a Bernstein-gamma function W_ϕ is the Mellin transform of a positive random variable. Therefore, there exists a positive Y_ϕ such that

$$W_\phi(z) = \mathbb{E}\big[Y_\phi^{z-1}\big]. \tag{9.1}$$

Moreover, by [PS21, Theorem 6.0.1 (1)], Y_ϕ is characterised by its entire moments via, for each $n \geq 1$,

$$\mathbb{E}\big[Y_\phi^n\big] = W_\phi(n) = \prod_{k=1}^n \phi(k).$$

9.2 Weierstrass product

Another well-known representation of the gamma function is due to Weierstrass: for $z \in \mathbb{C} \setminus \{-1, -2, \dots\}$,

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^\infty \frac{k}{k+z} e^{z/k},$$

where γ is the Euler-Mascheroni constant

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right).$$

This result is available, for example, as [Vio16, Theorem 6.3]. The generalisation to Bernstein-gamma functions is due to [PS21, Theorem 6.0.1 (2)]: at least on $\mathtt{Re}(z) > 0$,

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^\infty \frac{\phi(k)}{\phi(k+z)} e^{\phi'(k) z/\phi(k)},$$

where

$$\gamma_\phi := \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} - \ln(\phi(n)) \right)$$

9.3 Malmstèn formula

The next identity that we consider is the so-called Malmstèn formula, see [EMOT53, (1) on p.21], on $\mathtt{Re}(z) > -1$,

$$\log_0(\Gamma(z+1)) = \int_0^\infty \left(\frac{e^{-zy} - 1 - z(e^{-y} - 1)}{e^y - 1} \right) \frac{\mathrm{d}y}{y}.$$

The version for Bernstein-gamma functions was established in some cases on the real line by Hirsch and Yor [HY13, Theorem 3.1] and Berg [Ber07, Theorem 2.2]. Following Patie and Savov [PS18, Theorem 4.17.1] for $\texttt{Re}(z) > -1$, we have

$$\log_0(W_\phi(z+1)) = z\ln(\phi(1)) + \int_{[0,\infty)} \left(\frac{e^{-zy} - 1 - z(e^{-y} - 1)}{e^y - 1}\right)\frac{k_\phi(\mathrm{d}y)}{y},$$

where the measure $k_\phi(\mathrm{d}y)$ is defined in [PS18, Theorem 4.7 (4.15)] and, from [PS18, (5.37)], have Laplace transforms, for $\lambda > 0$,

$$\int_{[0,\infty)} e^{-\lambda y} k_\phi(\mathrm{d}y) = \frac{\phi'(\lambda)}{\phi(\lambda)}.$$

Let ξ be a Lévy process. When we work with Bernstein functions $\kappa_{q,\pm}(z)$ which are the Laplace exponents of the ladder process of ξ, that is in the notation of (3.3),

$$\kappa_{q,\pm}(z) := \kappa(q, z),$$

in Chapter V, we obtain a new probabilistic representation of the associated measures $k_{q,\pm}(\mathrm{d}y)$ via the harmonic potential of ξ:

$$k_{q,\pm}(\mathrm{d}y) \overset{(25.8)}{=} y\int_0^\infty e^{-qt}\frac{\mathbb{P}(\xi_t \in \pm\mathrm{d}y)}{t}.$$

See Lemma 24.1 for more information.

10 Stirling asymptotics

The first-order asymptotics of the factorial

$$n! = \Gamma(n+1) \overset{\infty}{\sim} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \frac{\sqrt{2\pi n}}{e}\exp\left(\int_1^n \ln(y)\mathrm{d}y\right)$$

were obtained first by De Moivre with the exact constant $\sqrt{2\pi}$ due to Stirling with both works published in 1730, see the historical note by Pearson [Pea24]. Following these results, it is standard to refer to their generalisation to the gamma and related functions as *Stirling asymptotics.*

Concerning Bernstein-gamma functions, the most convenient expression for their Stirling asymptotics is obtained by Barker and Savov in [BS21, Theorem 2.9]. Previous results have been derived by Patie and Savov [PS18], and when considering z on the real line, by Webster [Web97, Theorem 6.3] and Patie and Savov [PS21, Theorem 5.0.1], which in particular provide that, for $x > 0$ and some $C_\phi > 0$,

$$W_\phi(x+1) \overset{\infty}{\sim} \sqrt{C_\phi x}\exp\left(\int_1^x \ln(\phi(y))\mathrm{d}y\right),$$

similarly to the simple factorial case.

The precise result from [BS21, Theorem 2.9], states that, for $\texttt{Re}(z) > 0$,

$$W_\phi(z) = \frac{\phi^{1/2}(1)}{\phi(z)\phi^{1/2}(z+1)}e^{L_\phi(z)}e^{-E_\phi(z)},$$

where

$$L_\phi(z) := \int_{1\to z+1} \log_0(\phi(w))\mathrm{d}w,$$

with $\int_{1\to z+1}$ being any contour connecting 1 to $z+1$ lying in the domain of analyticity of $\log_0(\phi)$, and

$$\begin{aligned} \mathrm{P}(u) &:= (u-\lfloor u\rfloor)(1-(u-\lfloor u\rfloor)), \\ E_\phi(z) &:= \frac{1}{2}\int_1^\infty \mathrm{P}(u)\left(\log_0\left(\frac{\phi(u+z)}{\phi(u)}\right)\right)'' \mathrm{d}u. \end{aligned}$$

The term L_ϕ captures the main behaviour, but although explicit it may be difficult to analyse it for complex z. Under the condition of positive increase, see **(H)**, in Chapter IV, we provide such estimates in Theorem 19.3 and Corollary 19.5.

11 Formal definition and analytic properties

We start by defining the class $\mathcal{P}$ of *positive definite functions*, following [Jac01, definition 3.5.3], as

$$\mathcal{P} = \{M : i\mathbb{R} \to \mathbb{C} : \forall n \in \mathbb{N}, \sum_{j,k=1}^{n} M(s_j - s_k)z_j\bar{z}_k \geq 0 \text{ for } s_1,\dots,s_n \in i\mathbb{R}, z_1,\dots,z_n \in \mathbb{C}\}. \tag{11.1}$$

A probabilistic characterisation, known as Bochner theorem, see [Sat99, Proposition 2.5 (i)] or [Jac01, Theorem 3.5.7], is that $\psi \in \mathcal{P}$, $\psi(0)$, and ψ is continuous at $z=0$, if and only if there exists a random variable X_ψ such that, at least for $z \in i\mathbb{R}$,

$$\mathbb{E}\left[e^{zX_\psi}\right] = \psi(z).$$

Let us define the class of *shifted positive definite functions* as

$$\widetilde{\mathcal{P}} = \{M : z \mapsto M(z+1) \in \mathcal{P}\}.$$

Therefore, if $\varphi \in \widetilde{\mathcal{P}}$, we can represent it as

$$\varphi(z+1) = \mathbb{E}\left[e^{zX_\varphi}\right] = \mathbb{E}\left[\left(e^{X_\varphi}\right)^z\right],$$

so we see that

$$\text{if } \varphi \in \widetilde{\mathcal{P}}, \text{then } \varphi \text{ is the Mellin transform of the positive random variable } e^{X_\varphi}.$$

Setting $\mathcal{B}$ the class of Bernstein functions, we define Bernstein-gamma functions as the elements of

$$\mathcal{W}_{\mathcal{B}} := \left\{W \in \widetilde{\mathcal{P}} \colon W(1) = 1, W(z+1) = \phi(z)W(z) \text{ on } \mathtt{Re}(z) > 0, \text{ for some } \phi \in \mathcal{B}\right\}.$$

We summarise in the following theorem key analytic properties of W_ϕ. To do this recall the defined in (4.11) quantities

$$\begin{aligned} \mathfrak{a}_\phi &:= \inf\left\{u < 0 : \phi \text{ is analytic on } \mathbb{C}_{(u,\infty)}\right\} \in [-\infty, 0] \\ \mathfrak{u}_\phi &:= \sup\{u \in [\mathfrak{a}_\phi, 0] : \phi(u) = 0\} \in [-\infty, 0], \end{aligned}$$

$$\overline{\mathfrak{a}}_\phi := \max\{\mathfrak{a}_\phi, \mathfrak{u}_\phi\} \in [-\infty, 0].$$

Also, let us define the classes of analytic and meromorphic functions,

$$\mathtt{A}_{(a,b)} := \{f : f \text{ is analytic on } \mathbb{C}_{(a,b)}\}, \quad \text{and} \quad \mathtt{M}_{(a,b)} := \{f : f \text{ is meromorphic on } \mathbb{C}_{(a,b)}\}.$$

Then we have the following.

Theorem 11.1. *Let ϕ be a Bernstein function.*

1. *[PS21, Theorem 6.0.1 (3)] There exists a unique Bernstein-gamma function $W_\phi \in \mathcal{W}_{\mathcal{B}}$, associated with ϕ, that is a solution of*

$$W_\phi(z+1) = \phi(z)W_\phi(z) \text{ on } \mathtt{Re}(z) > 0, \quad W_\phi(1) = 1, \quad W_\phi \in \mathcal{W}_{\mathcal{B}}.$$

2. *[PS18, Theorem 4.1 (1)] It holds that*

$$W_\phi \in \mathtt{A}_{\left(\overline{\mathfrak{a}}_\phi, \infty\right)} \cap \mathtt{M}_{\left(\mathfrak{a}_\phi, \infty\right)}$$

and W_ϕ is zero-free on $\mathbb{C}_{\left(\mathfrak{a}_\phi, \infty\right)}$.

Further information on the on the analytic properties W_ϕ, also on its poles and respective residues, is available in [PS18, Theorem 4.1].

11.1 Associated functional equation

It turns out, see [Jac01, Theorem 3.6.16], that a function Ψ is the characteristic function of Lévy process ξ if and only if

$$z \mapsto \mathbb{E}\left[e^{z\xi_1}\right] = e^{\Psi(z)} \in \mathcal{P}, \quad \text{and} \quad \Psi(z) \text{ is continuous.} \tag{11.2}$$

Following standard notion, see [Jac01, Definition 3.6.5], the latter is the definition of

$$-\Psi \text{ is a } \textit{continuous negative definite function}, \text{ which we note } \Psi \in \overline{\mathcal{N}}.$$

Let $\Psi : i\mathbb{R} \to \mathbb{C}$ is an element of $\overline{\mathcal{N}}$. As explained, this is equivalent to the condition that Ψ is the characteristic exponent of a Lévy process, so it has a Lévy-Khintchine representation, see (2.3),

$$\Psi(z) = -q + \gamma z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{zx} - 1 - zx\mathbb{1}_{\{|x| \le 1\}}\right)\Pi(\mathrm{d}x).$$

We are then interested in solving

$$\mathcal{M}_\Psi(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_\Psi(z), \quad \mathcal{M}_\Psi(1) = 1, \tag{11.3}$$

on at least the domain $i\mathbb{R} \setminus (\mathcal{Z}_0(\Psi) \cup \{0\})$, where we set $\mathcal{Z}_0(\Psi) := \{z \in i\mathbb{R} : \Psi(-z) = 0\}$. Let the associated one-dimensional Wiener-Hopf factorisation of Ψ, see (3.3), be, for $z \in i\mathbb{R}$,

$$\Psi(z) = -\phi_+(-z)\phi_-(z).$$

Then, Patie and Savov [PS18] showed that a solution of (11.3) can be expressed in terms of the Bernstein-gamma functions $W_{\phi_\pm}$.

Theorem 11.2. *[PS18, Theorem 2.1] Let $\Psi \in \overline{\mathcal{N}}$ and $\phi_\pm$ be its Wiener-Hopf factors. Then, the mapping M_Ψ defined by*

$$M_\Psi(z) := \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \tag{11.4}$$

satisfies the recurrence relation (11.3) *at least on $i\mathbb{R} \setminus (\mathcal{Z}_0(\Psi) \cup \{0\})$. Moreover, with the notation $\mathfrak{a}_\pm := \mathfrak{a}_{\phi_\pm}$ and $\overline{\mathfrak{a}}_\pm := \overline{\mathfrak{a}}_{\phi_\pm}$ from* (4.11)*, we have that*

$$M_\Psi \in \mathtt{A}_{\left(\mathfrak{a}_+ \mathbb{1}_{\{\overline{\mathfrak{a}}_+=0\}}, 1-\overline{\mathfrak{a}}_-\right)}.$$

A direct consequence is, since the Mellin transform of I_ξ solves (11.3), that

$$\mathcal{M}_{I_\Psi}(z) = \mathbb{E}\left[I_\xi^{z-1}\right] = M_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z),$$

which opens the route of exploring the properties of the exponential functional via understanding the functions Bernstein-gamma $W_{\phi_\pm}$. Another context in which equation (11.3) is relevant is the study of some non-self-adjoint Markov semigroups as in [PS17], [PS21], [PSZ19] and [PZ17].

12 Explicit examples

Many examples are available from Bartholmé and Patie [BP21, Section 1.2]. For completeness of the thesis, we also provide them in Section 12 of the thesis.

13 Bivariate Bernstein-gamma functions

We have seen that for each Bernstein function ϕ, we can define a Bernstein-gamma function W_ϕ as the solution to an equation like (8.3), that is, for $\mathtt{Re}(z) > 0$,

$$W_\phi(z+1) = \phi(z) W_\phi(z), \quad W_\phi(1) = 1.$$

We recall that bivariate Bernstein functions are exactly the Laplace exponents of bivariate subordinators, which we presented in Section 3. In [BS21], Barker and Savov showed that the equation, on $\mathtt{Re}(\zeta) \geq 0$ and $\mathtt{Re}(z) > 0$,

$$W_\kappa(\zeta, z+1) = \kappa(\zeta, z) W_\kappa(\zeta, z) \quad W_\kappa(\zeta, 1) = 1,$$

has a unique analytic solution on $\mathbb{C}^2_{(0,\infty)}$ such that for any $q \geq 0$, the function $W_\kappa(q, \cdot)$ is the Mellin transform of a positive random variable. The function W_κ is called a *bivariate Bernstein-gamma function.* Its analytic properties, Weierstrass product representation, and Stirling asymptotics are available in [BS21]. In the case where $\kappa_\pm$ are the Wiener-Hopf factors the characteristic exponent Ψ of a Lévy process ξ, see (3.3), in Section 24 of Chapter V, we understand in depth the link between the q-derivatives of $W_{\kappa_\pm}(q, z)$ and the probabilistic properties of ξ via its q-potential measures.

Chapter IV

Asymptotics for densities of exponential functionals of subordinators

14 Introduction and motivation

In this chapter, we predominantly work with exponential functionals of killed processes, i.e. with

$$I_{\xi,q} := \int_0^{\mathbf{e}_q} e^{-\xi_s} \mathrm{d}s$$

for $\mathbf{e}_q \sim Exp(q)$ independent of ξ. We recall that we presented these objects in Section 5. These random variables are well defined for $q > 0$ and if $q = 0$, we must assume that ξ drifts to infinity, see Theorem 4.1. In Section 5.3 we outlined the work done on the tail asymptotics, and even further on the density $f^{(n)}_{I_{\xi,q}}$, of $I_{\xi,q}$. We present in detail the results from Theorem 5.1.3, as they appear in [MS23].

First, let us recall that we introduced the class of subordinators in Section 2.1, in particular, as in (2.5), for these processes, the Laplace exponent ϕ is well defined on $\mathtt{Re}(z) \geq 0$ by

$$\mathbb{E}\left[e^{-zX_t}\right] = e^{-t\phi(z)},$$

and

$$\phi(z) = q + \mathtt{d}z + \int_{(0,\infty)} \left(1 - e^{-zy}\right)\mu(\mathrm{d}y) = q + \mathtt{d}z + z\int_0^\infty e^{-zy}\bar{\mu}(y)\mathrm{d}y, \tag{14.1}$$

where $\mathtt{d} \geq 0$ is the drift term, the Lévy measure $\mu(\mathrm{d}y)$ supported on $(0,\infty)$ is such that

$$\int_{(0,\infty)} \min\{y,1\}\mu(\mathrm{d}y) = \int_0^1 \bar{\mu}(w)\mathrm{d}w < \infty, \quad \text{where} \quad \bar{\mu}(y) := \int_{(y,\infty)} \mu(\mathrm{d}s). \tag{14.2}$$

For this chapter, we mean potentially killed subordinator if we refer to subordinator only.

Next, let us note that if $\mathtt{d} > 0$, [PS18, Theorem 2.4.2] shows that the support of $I_{\xi,q}$ is bounded, and even more precisely it is $[0, 1/\mathtt{d}]$. Therefore, the question of finding its asymptotics at infinity is trivial. That is why **until the end of this chapter, we consider ξ to be a potentially killed driftless subordinator with Laplace exponent ϕ.**

The first major results on the tail asymptotics of $I_{\xi,q}$ were obtained by Haas and Rivero in [HR12, Section 3] under the assumption of positive increase:

Assumption (H): ξ is a driftless subordinator, i.e. $\mathtt{d} = 0$ in (14.1), and its Lévy measure has positive increase, that is

$$\liminf_{x\to 0+} \frac{\int_0^{2x} \bar{\mu}(y)\mathrm{d}y}{\int_0^x \bar{\mu}(y)\mathrm{d}y} > 1 \iff \limsup_{x\to\infty} \frac{x\phi'(x)}{\phi(x)} < 1. \tag{14.3}$$

We say that a Bernstein function has positive increase if it is the Laplace exponent of a subordinator with positive increase. Then, under **(H)**, [HR12, Proposition A.1] provides and for each $q \geq 0$, the asymptotics of $I_{\xi,q}$ on a log-scale,

$$-\ln(\mathbb{P}(I_{\xi,q} > x)) \stackrel{\infty}{\sim} \int_{\phi_*(1)}^{x} \frac{\varphi_*(y)}{y}\mathrm{d}y, \quad \text{where} \quad \phi_*(z) := \frac{z}{\phi(z)}, \quad \varphi_* := \phi_*^{-1}. \tag{14.4}$$

We prove in Corollary 17.2 that under the same conditions, the asymptotics on the real scale are

$$\mathbb{P}(I_{\xi,q} > x) \stackrel{\infty}{\sim} C_\phi \frac{x\sqrt{\varphi_*'(x)}}{\varphi_*(x)} e^{-\int_{\phi_*(1)}^{x} \frac{\varphi_*(y)}{y}\mathrm{d}y}, \tag{14.5}$$

where the exact value of the constant C_ϕ is available in (17.1). Moreover, the main result of this chapter, Theorem 17.1, implies that, for each $n \geq 0$,

$$f^{(n)}_{I_{\xi,q}}(x) \stackrel{\infty}{\sim} \frac{C_\phi \varphi_*^n(x)\sqrt{\varphi_*'(x)}}{x^n} e^{-\int_{\phi_*(1)}^{x} \frac{\varphi_*(y)}{y}\mathrm{d}y}. \tag{14.6}$$

For the case $q = 0$, the equivalences (14.5) and (14.6) for $n = 0$ were also independently obtained by Haas in [Haa24] shortly after [MS23]. We comment on the similarities and differences of the two works in Remark 17.3 of the thesis.

There are various scenarios in which the exponential functionals of subordinators may play a crucial role. For example, in the study of fragmentation processes, see Section 7.2, where the time to dust is exactly I_ξ. Another example is the study of Yaglom limits of self-similar Markov processes, see [HR12]. More generally, we recall that, by [PS18, Theorem 2.22], every exponential functional of a Lévy process η can be factorised as an independent product of the exponential functional of its ladder height process and another positive random variable, that is,

$$I_{\eta,q} = I_{\kappa_+,q} \times Y, \tag{14.7}$$

so the analysis of $I_{\eta,q}$ can be reduced to the simpler objects $I_{\kappa_+,q}$ and Y. As an illustration of this, we show in Corollary 17.6 that analyticity in some cone of the density of $I_{\kappa_+,q}$, implies analyticity of the density of $I_{\eta,q}$. We further prove in Theorem 17.1 that under one more mild condition apart from (14.3) **(H)**, $f_{I_{\xi,q}}$ is indeed analytic in a cone of the complex plane. This property can also have deep implications similar to the analyticity in a cone has had for the spectral expansions of non-selfadjoint Markov semigroups derived in [PS17].

The obtained results on Bernstein-gamma functions are interesting on their own, serving as valuable contributions within the broader context of special functions. In (19.9) we obtain a convenient form of the Stirling asymptotics for

$$\mathcal{M}_{I_\xi}(z) = \mathbb{E}\big[I_\xi^{z-1}\big] = \frac{\Gamma(z)}{W_\phi(z)}.$$

In order to analyse it further, in Theorem 19.3, we obtain how Bernstein functions act on rays in the positive complex half-plane, that is, $\mathbb{C}_{(0,\infty)}$, which pass through the origin. Under one more condition, in addition to **(H)**, by Corollary 19.5, we obtain exponential decay of $|\Gamma(z)/W_\phi(z)|$ along complex lines in $\mathbb{C}_{(0,\infty)}$.

15 Preliminaries

We state again that for a potentially killed at rate q subordinator ξ, in (14.1), we defined its Laplace exponent, for $\mathtt{Re}(z) \geq 0$, as

$$\phi(z) := \log_0\!\big(\mathbb{E}\big[e^{zX_1}\big]\big) = q + \mathtt{d}z + \int_{(0,\infty)} \big(1 - e^{-zy}\big)\mu(\mathrm{d}y) = q + \mathtt{d}z + z\int_0^\infty e^{-zy}\bar{\mu}(y)\mathrm{d}y.$$

We recall that ϕ is a Bernstein function, and the Bernstein functions are in bijection with the Laplace exponents of potentially killed subordinators. Unless ξ is identically 0 and $q = 0$, ξ drifts to infinity, so Theorem 4.1 ensures that $I_{\xi,q}$ is a finite a.s. random variable.

According to Theorem 5.2, if $\mathtt{d} = 0$, the law of $I_{\xi,q}$ is infinitely differentiable, and we denote its density by $f_{I_{\xi,q}}$. Then, [PS18, (2.24)] provides the following Mellin-Barnes representation, for all $n \geq 0$ and $a, x > 0$,

$$f^{(n)}_{I_{\xi,q}}(x) = \frac{(-1)^n}{2\pi i}\int_{\mathtt{Re}(z)=a} x^{-z-n}\frac{\Gamma(z+n)}{W_\phi(z)}\mathrm{d}z. \tag{15.2}$$

We emphasise that in the notation of [PS18] we have that $\phi_+ \equiv \phi$, $\phi_- \equiv 1$, $d_+ = d_- = \mathtt{d} = 0$, and $\bar{\mathfrak{a}}_- = -\infty$, so (2.24) therein leads exactly to (15.2). Furthermore, although not explicitly stated, [PS18] provides a similar representation of the tail of $I_{\xi,q}$ as well: by [PS18, (7.15)] and the Mellin inversion formula,

$$\mathbb{P}(I_{\xi,q} > x) = \frac{1}{2\pi i}\int_{\mathtt{Re}(z)=a} x^{-z}\frac{1}{z}\frac{\Gamma(z+1)}{W_\phi(z+1)}\mathrm{d}z = \frac{1}{2\pi i}\int_{\mathtt{Re}(z)=a} x^{-z}\frac{1}{\phi(z)}\frac{\Gamma(z)}{W_\phi(z)}\mathrm{d}z. \tag{15.3}$$

Next, we define, for $\mathtt{Re}(z) > 0$,

$$\phi_*(z) := \frac{z}{\phi(z)}, \tag{15.4}$$

which is well defined, since it can be easily verified that $\phi(z)$ is zero-free in this region, see (A.4). Also,

$$\lim_{x\to 0+}\phi_*(x) = \frac{\mathbb{1}_{\{q=0\}}}{\phi'(0+)} \in [0,\infty)$$

since, from Proposition A.1.1, it holds true that $\phi'(0+) \in (0,\infty]$. Next, from Proposition A.1.2, we have that ϕ_* is increasing on $x > 0$ and $\lim_{x\to\infty}\phi_*(x) = \infty$ because we are working in the case $\mathtt{d} = 0$. This, in turn, allows us to define the inverse of ϕ_*

$$\varphi_* := \phi_*^{-1} : Dom(\varphi_*) = \left(\frac{1}{\phi'(0+)}\mathbb{1}_{\{q=0\}}, \infty\right) \to (0,\infty). \tag{15.5}$$

16 Methodology

Our approach is based on the application of a saddle point method to (15.2), which requires the optimal choice of the contour of integration to depend on x, that is, $\texttt{Re}(z) = a = a(x)$. While this choice is not hard to guess from the available Stirling asymptotics for $\Gamma(z)/W_\phi(z)$, the information on the latter in the literature is not sharp enough to control all terms required for the derivation of the asymptotics. In Section 19 we provide the required analytic results.

17 Main results

The following is the main result of this chapter.

Theorem 17.1. *Under **(H)** , for each $n \geq 0$ and $q \geq 0$,*

$$f^{(n)}_{I_{\xi,q}}(x) \stackrel{\infty}{\sim} C_\phi \frac{\varphi^n_*(x)\sqrt{\varphi'_*(x)}}{x^n} e^{-\int_{\phi_*(1)}^{x} \frac{\varphi_*(y)}{y}\mathrm{d}y}, \quad \text{with } C_\phi = \frac{(-1)^n e^{-T_{\phi_*}}}{\sqrt{2\pi\phi_*(1)}}, \tag{17.1}$$

and

$$T_{\phi_*} = \int_1^\infty (u - \lfloor u \rfloor)(1-(u-\lfloor u \rfloor))\left(\frac{1}{u^2} - \left(\frac{\phi'(u)}{\phi(u)}\right)^2 + \frac{\phi''(u)}{\phi(u)}\right)\mathrm{d}u \in (-\infty, \infty),$$

where $\lfloor \cdot \rfloor$ stands for the floor function.

If in addition $\limsup_{x\to 0+} \bar{\mu}(2x)/\bar{\mu}(x) < 1$, then there exists $\varepsilon > 0$ such that $f_{I_{\xi,q}}$ is analytic in the cone $\{z \in \mathbb{C} : \texttt{Re}(z) > 0 \text{ and } |\arg z| < \varepsilon\}$.

Corollary 17.2. *Under the conditions of Theorem 17.1,*

$$\mathbb{P}(I_{\xi,q} > x) \stackrel{\infty}{\sim} C_\phi \frac{x\sqrt{\varphi'_*(x)}}{\varphi_*(x)} e^{-\int_{\phi_*(1)}^{x} \frac{\varphi_*(y)}{y}\mathrm{d}y}. \tag{17.2}$$

Next, we provide some consequences from the main result, which may are useful for direct applications.

Corollary 17.4. *Let ξ be a potentially killed non-decreasing compound Poisson process for which $\int_0^1 \mu(\mathrm{d}v)/v < \infty$. Then*

$$f^{(n)}_{I_{\xi,q}}(x) \stackrel{\infty}{\sim} Ce^{-\phi(\infty)x}, \tag{17.3}$$

with

$$C = (-1)^n \phi^n(\infty)\sqrt{\phi(\infty)} e^{\phi(\infty)\phi_*(1) + \int_0^\infty \int_{\phi_*(1)}^\infty e^{-\varphi_*(y)v}\mathrm{d}y\mu(\mathrm{d}v)} \frac{e^{-T_{\phi_*}}}{\sqrt{2\pi\phi_*(1)}}.$$

If $\int_0^1 \mu(\mathrm{d}v)/v = \infty$, then

$$f^{(n)}_{I_{\xi,q}}(x) \stackrel{\infty}{\sim} Ce^{-\phi(\infty)(x+\mathrm{o}(x))} \quad \text{with } C = (-1)^n \phi^n(\infty)\sqrt{\phi(\infty)} e^{\phi(\infty)\phi_*(1)} \frac{e^{-T_{\phi_*}}}{\sqrt{2\pi\phi_*(1)}}. \tag{17.4}$$

Finally, in all cases, the density $f_{I_{\xi,q}}$ is analytic in the cone $\{z \in \mathbb{C} : \texttt{Re}(z) > 0$ and $|\arg z| < \pi/2\}$.

Next, using the factorisation (14.7),

$$I_{\eta,q} = I_{\phi_+,q} \times Y,$$

we show that the analyticity of $f_{I_{\kappa_+,q}}$ implies analyticity of $f_{I_{\eta,q}}$.

Corollary 17.6. *Let Ξ be a general Lévy process such that $I_\Xi < \infty$ almost surely. Assume that the Bernstein function $\phi_{q,+}$, stemming from the Wiener-Hopf factorisation* (3.3) *of Ξ, satisfies **(H)**. Then the density of $I_{\Xi,q}$ is analytic at least on the same cone as the density of $I_{\phi_{q,+}}$.*

18 Examples

In this chapter explicit examples are presented for the results of Theorem 17.1 and Corollary 17.2 in the cases where ϕ is regularly varying, and more specifically, for stable and gamma subordinators.

19 Needed new results on Bernstein-gamma functions

We recall that one of the reasons Bernstein-gamma functions appear naturally in the study of exponential functionals is that, for $\texttt{Re}(z) > 0$,

$$\mathbb{E}\left[I_{\xi,q}^{z-1}\right] = \frac{\Gamma(z)}{W_\phi(z)}. \tag{19.2}$$

The next result describes the Stirling asymptotics of the latter quantity.

Lemma 19.1. *Let ϕ be a Bernstein function. Then, for* $\texttt{Re}(z) > 0$,

$$\frac{\Gamma(z)}{W_\phi(z)} = \frac{\phi_*^{1/2}(1)}{\phi_*(z)\phi_*^{1/2}(z+1)} e^{L_{\phi_*}(z)} e^{-E_{\phi_*}(z)}, \tag{19.6}$$

where, for $\texttt{Re}(z) > 1$,

$$\begin{aligned} L_{\phi_*}(z-1) &:= \int_{1\to\texttt{Re}(z)\to\texttt{Re}(z)+i\texttt{Im}(z)} \log_0(\phi_*(w))\mathrm{d}w \\ &= \int_1^{\texttt{Re}(z)} \ln(\phi_*(w))\mathrm{d}w - \int_0^{\texttt{Im}(z)} \arg(\phi_*(\texttt{Re}(z)+iw))\mathrm{d}w + i\int_0^{\texttt{Im}(z)} \ln|\phi_*(\texttt{Re}(z)+iw)|\mathrm{d}w \\ &=: G_{\phi_*}(\texttt{Re}(z)) - A_{\phi_*}(z) + iU_{\phi_*}(z). \end{aligned} \tag{19.7}$$

Moreover, uniformly in $\texttt{Im}(z)$,

$$T_{\phi_*} := \lim_{\texttt{Re}(z)\to\infty} E_{\phi_*}(z) = \int_1^\infty \mathrm{P}(u)\left(\frac{1}{u^2} - \left(\frac{\phi'(u)}{\phi(u)}\right)^2 + \frac{\phi''(u)}{\phi(u)}\right)\mathrm{d}u. \tag{19.8}$$

Finally, as $\texttt{Re}(z) \to \infty$,

$$\frac{\Gamma(z)}{W_\phi(z)} = (1+\mathrm{o}(1))\frac{\phi_*^{1/2}(1)}{\phi_*^{1/2}(z+1)} e^{L_{\phi_*}(z-1)} e^{-T_{\phi_*}}, \tag{19.9}$$

where the asymptotic relation $\mathrm{o}(1)$ *holds uniformly in* $\texttt{Im}(z)$.

Let us define, for $\pi/2 \geq \theta > 0$,

$$\mathcal{B}_{exp}(\theta) := \left\{ \phi \in \mathcal{B} : \forall a > 0, \forall \epsilon \in (0,\theta) : \lim_{|b|\to\infty} e^{(\theta-\epsilon)|b|} \left| \frac{\Gamma(a+ib)}{W_\phi(a+ib)} \right| = 0 \right\}. \tag{19.10}$$

Our next result aims to obtain more information about which Bernstein functions belong to the class $\mathcal{B}_{exp}(\theta)$. For this purpose, we first give some information on $\arg \phi_*$ as it will drive the asymptotics of (19.7) via A_{ϕ_*}.

Theorem 19.3. *Let ϕ be a Bernstein function under **(H)**. Then, for any $\eta > 0$, there exist $a_\eta > 0$ and $C_\eta > 0$ such that if $t \geq \eta$ and $a \geq a_\eta$, we have that*

$$\arg(a(1+it)) - \arg(\phi(a(1+it))) = \arg \phi_*(a(1+it)) \geq \frac{C_\eta}{t}, \tag{19.11}$$

or equivalently,

$$\arg(\phi(a(1+it))) \leq \arctan(t) - \frac{C_\eta}{t}.$$

Moreover, if in addition $\limsup_{x\to 0} \bar{\mu}(2x)/\bar{\mu}(x) < 1$, for any $\eta > 0$, there exist $a'_\eta > 0$ and $\varepsilon_\eta \in (0, \pi/2)$ such that, for $t \geq \eta$ and $a \geq a'_\eta$,

$$\arg(a(1+it)) - \arg(\phi(a(1+it))) = \arg \phi_*(a(1+it)) \geq \varepsilon_\eta, \tag{19.12}$$

or equivalently,

$$\arg \phi(a(1+it)) \leq \arctan(t) - \varepsilon_\eta.$$

Theorem 19.3 is very useful, since it gives an estimate for the arguments of Bernstein functions with conditions on the Lévy measure, which is often assumed to be known explicitly, or at least its asymptotics are given a priori.

Next, we transfer these results to the asymptotics of $\Gamma(z)/W_\phi(z)$.

Corollary 19.5. *Let ϕ be a Bernstein function under **(H)** and $\limsup_{x\to 0+} \bar{\mu}(2x)/\bar{\mu}(x) < 1$. Then there exist $\varepsilon > 0$ and constants a_0, $t_0 > 0$ such that, for any $a \geq a_0$, and $|b| \geq at_0$,*

$$A_{\phi_*}(a+ib) \geq \varepsilon|b| - \varepsilon a t_0. \tag{19.13}$$

As a result, $\phi \in \mathcal{B}_{exp}(\varepsilon)$. This, in particular, is true if $\bar{\mu}(y) \in \mathcal{R}_\alpha$ with $\alpha \in (0,1)$, that is, $\bar{\mu}(y) \overset{0}{\sim} y^{-\alpha}\ell(y)$ where $\ell \in SV_0$ is a slowly varying function at 0.

The next result contrasts with the assumptions of Corollary 19.5 and shows that for exponential decay along complex lines, there is much room for improvement.

Lemma 19.8. *If ϕ is a Bernstein function such that $\phi(\infty) < \infty$, then $\phi \in \mathcal{B}_{exp}(\pi/2)$.*

20 Proofs

First, in this section of the thesis are presented the proofs of Theorem 17.1, Corollary 17.2, Corollary 17.4.

Then it continues with the proofs of the results from Section 19, Lemma 19.1, Theorem 19.3, and Lemma 19.8.

Chapter V

Bivariate Bernstein-gamma functions, potential measures, and asymptotics of exponential functionals of Lévy processes

21 Introduction and motivation

In this chapter we continue with the analysis of exponential functionals of Lévy processes but we will concentrate on such on deterministic horizon

$$I_\xi(t) := \int_0^t e^{-\xi_s}\mathrm{d}s. \tag{21.1}$$

We recall that we presented these objects and the available results concerning them in Section 6.

The most advanced knowledge for exponential functionals of Lévy processes on deterministic horizon is available for the asymptotic behaviour of distributional quantities of $I_\xi(t)$, as $t \to \infty$, provided $I_\xi := I_\xi(\infty) = \infty$ almost surely. These results are predominantly driven by applications, especially in the area of branching processes in Lévy random environments, see Section 7.1. In this chapter of the thesis are obtained new results in this case. More specifically, under the assumptions $\mathbb{E}[\xi_1] \in (-\infty, 0)$ and $\mathbb{P}(\xi_1 > t)$ is regularly varying at infinity with index $\alpha > 1$, we show that the measures

$$\frac{y^{-a}\mathbb{P}(I_\xi(t) \in \mathrm{d}y)}{\mathbb{P}(\xi_1 > t)}, \quad a \in (0,1), \tag{21.2}$$

converge weakly, as $t \to \infty$, to a finite positive measure supported by $(0,\infty)$. Via Mellin inversion, we give a semi-explicit analytic description of the cumulative distribution function of the limit measure, which also yields the asymptotic behaviour of $\mathbb{E}\left[I_\xi^{-a}(t)\right]$, see Theorem 23.1 and the ensuing corollaries. The latter allows us to derive a neat probabilistic representation of the normed limit measure which turns out to be the law of a curious generalisation of the main product factorisation for classical exponential functionals, see Theorem 23.5. Using it, we obtain novel analytic properties for the limit measure, such as the existence and smoothness of its density. Also, dropping the requirement for regular variation of the upper tail and finiteness of the expectation, we obtain general upper bounds for the decay of quantities of the type

$$\mathbb{E}[F(I_\xi(t))], \quad \text{as } t \to \infty,$$

which frequently arise in the study of random processes in random environments as we described earlier in Section 7.1. The rate of decay depends on the finiteness of an explicit integral criterion, see Theorem 23.6.

One of the main ingredients in the proof of the aforementioned results is the new information, which is of independent interest, that we obtain on the functional-analytic properties of the bivariate Bernstein-gamma functions. It is known that the bivariate Bernstein-gamma functions generated by the two Wiener-Hopf factors $\phi_{q,\pm}$, see (3.3), of the underlying Lévy process ξ, represent the Mellin transform of the classical exponential functional via the equation, at least when $\mathtt{Re}(z) \in (0,1)$,

$$\mathcal{M}_{I_{\xi,q}}(z) = \phi_-(q,0)\frac{\Gamma(z)}{W_{\phi_{q,+}}(z)}W_{\phi_{q,-}}(1-z).$$

Therefore by Mellin inversion, we can obtain information on $I_{\xi,q}$, and then in turn by Laplace transform to draw conclusions regarding $I_\xi(t)$.

We show that bivariate Bernstein-gamma functions related to the Wiener-Hopf factors of transient Lévy processes are n times differentiable at zero in the first variable if and only if an explicit integral criterion is finite, see Theorem 24.3. Moreover, we link the derivatives of these functions to the convolutions and the derivatives of the q-potential measures of the underlying Lévy process, see Lemma 24.1, thereby deriving some new results for as basic objects as these potentials. When the potential measure has a bounded density, we obtain upper bounds for the densities of the convolutions of the potential measure, see Theorem 24.6. Using these results, we derive universal and seemingly quite handy estimates for the derivatives of the Mellin transforms of the classical exponential functionals and their decay along complex lines, see Corollary 24.4.

The results on the derivatives of bivariate Bernstein-gamma functions play a crucial role in a convoluted Tauberian approach that we adopt in order to derive the weak convergence of the measures defined in (21.2). The method requires a particularly intricate analysis when the regular variation index α of $\mathbb{P}(\xi_1 > t)$ is an integer. Then we have to appeal to the de Haan theory, and in order to utilise it successfully, we have to resort to probabilistic considerations, which complement the analytical approach. The latter, in turn, reveals that one may have a more direct way to study the moments of the measure (21.2), but at the cost of a loss of explicitness and precision. A similar probabilistic approach has been adopted in the predecessor of our work, see [Xu21]. We discuss further our methodology in Section 22.3

22 Preliminaries, methodology, background, and literature review

22.1 Notation

We recall some basic notation. We use $\mathbb{C}$ to denote the complex plane with z representing a complex variable with a real part $\mathtt{Re}(z)$ and an imaginary part $\mathtt{Im}(z)$, and $\mathbb{R}_+$ for the set of non-negative real numbers, $[0,\infty)$. For a set $A \subset \mathbb{R}$, we employ $\mathbb{C}_A$ for the subset of the complex plane $\mathbb{C}_A := \{z \in \mathbb{C} : \mathtt{Re}(z) \in A\}$. Furthermore, unless otherwise stated, n, k, l will be positive integers. For any non-negative random variable X, we denote by $\mathcal{M}_X(z) := \mathbb{E}[X^{z-1}]$ its Mellin transform, which is defined at least on the complex line $\mathbb{C}_{\{1\}} = \{z \in \mathbb{C} : \mathtt{Re}(z) = 1\}$. If μ is a measure, we define for better readability $\mu(a,b) := \mu((a,b))$. Next, for a function f, we formally define its Laplace transform, for $q \geq 0$, as $\widehat{f}(q) := \int_0^\infty e^{-qt} f(t)\mathrm{d}t$. For asymptotic behaviour, we use the classical Landau notation and also $f \overset{a}{\sim} g$ to denote $\lim_{x\to a} f(x)/g(x) = 1$. We recall that a function ϕ is a Bernstein function if and only if it can be represented as

$$\phi(z) = \phi(0) + \mathtt{d}z + \int_0^\infty \left(1 - e^{-zy}\right)\mu(\mathrm{d}y), \quad z \in \mathbb{C}_{[0,\infty)}, \tag{22.1}$$

where $\phi(0) \geq 0$, $\mathtt{d} \geq 0$, and μ is a σ-finite positive measure such that $\int_0^\infty \min\{1, y\}\mu(\mathrm{d}y) < \infty$. Note that each Bernstein function is the Laplace exponent of a potentially killed subordinator with killing rate $\phi(0)$, linear drift $\mathtt{d}$ and jump measure μ. In the same fashion, a bivariate Bernstein function κ, given by, for $(\zeta, z) \in \mathbb{C}_{[0,\infty)} \times \mathbb{C}_{[0,\infty)}$,

$$\kappa(\zeta, z) = \kappa(0,0) + \mathtt{d}_1\zeta + \mathtt{d}_2 z + \int_0^\infty \int_0^\infty \left(1 - e^{-\zeta y_1 - z y_2}\right)\mu(\mathrm{d}y_1, \mathrm{d}y_2) \tag{22.2}$$

is the Laplace exponent of a potentially killed bivariate subordinator, where $\kappa(0,0)$, $\mathtt{d}_1$ and $\mathtt{d}_2$ are non-negative and have the same meaning as above, and μ is the bivariate jump measure, which satisfies $\int_0^\infty \int_0^\infty \min\{1, \sqrt{y_1^2 + y_2^2}\}\mu(\mathrm{d}y_1, \mathrm{d}y_2) < \infty$.

22.2 Bivariate Bernstein-gamma functions and the Mellin transform of exponential functionals

From [BS21, Definition 2.5, Theorem 2.8], we know that for any given bivariate Laplace exponent $\kappa \not\equiv 0$, the equation

$$W_\kappa(\zeta, z+1) = \kappa(\zeta, z)W_\kappa(\zeta, z), \qquad W_\kappa(\zeta, 1) = 1, \zeta \in \mathbb{C}_{[0,\infty)}, z \in \mathbb{C}_{(0,\infty)}, \tag{22.3}$$

has a unique bivariate holomorphic solution in $\mathbb{C}_{(0,\infty)} \times \mathbb{C}_{(0,\infty)}$. Moreover, for any $q \geq 0$, the function $W_\kappa(q, \cdot)$ is analytic on $\mathbb{C}_{(0,\infty)}$, and it is a Mellin transform of a positive random variable.

From now on, we work with the case where the bivariate Bernstein-gamma functions are based on the Wiener-Hopf factors of Lévy processes. We briefly present key facts, based on Section 3.

Let ξ be a Lévy process with Lévy-Khintchine exponent Ψ. Then, the characteristic exponent of its potentially killed at rate q version is

$$\Psi_q(z) = \log_0 \mathbb{E}\left[e^{z\xi_1}\right] = \Psi(z) - q = \gamma z + \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty} \left(e^{zy} - 1 - zy\mathbb{1}_{\{|y|\leq 1\}}\right)\Pi(\mathrm{d}y) - q, \tag{22.4}$$

where we use $\log_0$ for the principal branch of the complex logarithm, $\gamma \in \mathbb{R}$ and $\sigma^2 \geq 0$ are respectively the linear term and the Brownian component of ξ, Π is the Lévy measure of the process which satisfies $\int_{-\infty}^{\infty} \min\{1, y^2\}\Pi(\mathrm{d}y) < \infty$, and q is the killing rate.

For any potentially killed Lévy process, we have the Wiener-Hopf factorisation, see (3.3),

$$\Psi_q(z) = \Psi(z) - q = -\phi_+(q, -z)\phi_-(q, z) = -h(q)\kappa_+(q, -z)\kappa_-(q, z), \quad z \in i\mathbb{R}, \tag{22.5}$$

where, for $q \geq 0$,

$$h(q) := \exp\left(-\int_0^\infty \left(\frac{e^{-t} - e^{-qt}}{t}\right)\mathbb{P}(\xi_t = 0)\mathrm{d}t\right), \tag{22.6}$$

which in the case of a transient Lévy process ξ is non-zero by [Ber96, Theorem 12], and, for $q \geq 0$ and $\mathtt{Re}(z) \geq 0$,

$$\kappa_\pm(q, z) = c_\pm \exp\left(\int_0^\infty \int_{[0,\infty)} \left(\frac{e^{-t} - e^{-qt-zx}}{t}\right)\mathbb{P}(\pm\xi_t \in \mathrm{d}x)\mathrm{d}t\right) \tag{22.7}$$

are the bivariate Laplace exponents of the ascending/descending ladder processes which form bivariate subordinators, see Section 3. We note that the constants $c_\pm > 0$ depend on the choice of local times, which we can choose appropriately so that $c_+c_- = 1$. Furthermore, we can choose

$$\phi_+(q, z) = \kappa_+(q, z)/c_+, \quad \text{and} \quad \phi_-(q, z) = h(q)\kappa_-(q, z)/c_-, \tag{22.8}$$

i.e., for $q \geq 0$ and $\mathtt{Re}(z) \geq 0$,

$$\begin{aligned}
\phi_+(q, z) &= \exp\left(\int_0^\infty \int_{[0,\infty)} \left(\frac{e^{-t} - e^{-qt-zx}}{t}\right)\mathbb{P}(\xi_t \in \mathrm{d}x)\mathrm{d}t\right), \\
\phi_-(q, z) &= \exp\left(\int_0^\infty \int_{(0,\infty)} \left(\frac{e^{-t} - e^{-qt-zx}}{t}\right)\mathbb{P}(\xi_t \in -\mathrm{d}x)\mathrm{d}t\right).
\end{aligned} \tag{22.9}$$

We note that $h(q) \equiv 1$ provided that ξ is not a compound Poisson process (CPP), and that in this case $h(q) \equiv 1$ and $\phi_\pm \equiv \kappa_\pm/c_\pm$ are bivariate Laplace exponents as well. When $h(q) \not\equiv 1$, for any fixed $q \geq 0$, $\phi_{q,\pm}(z) := \phi_\pm(q, z)$ are Laplace exponents of univariate subordinators.

With the notation, for $q \geq 0$,

$$I_{\xi,q} := I_\xi(\mathbf{e}_q) = \int_0^{\mathbf{e}_q} e^{-\xi_s}\mathrm{d}s,$$

from [PS18, Theorem 2.4] we know that, at least for $z \in \mathbb{C}_{(0,1)}$,

$$\mathcal{M}_{I_{\xi,q}}(z) = \phi_-(q, 0)\frac{\Gamma(z)}{W_{\phi_{q,+}}(z)}W_{\phi_{q,-}}(1 - z) =: \phi_-(q, 0)M_\Psi(q, z), \tag{22.10}$$

where for the fixed q, the functions $W_{\phi_{q,\pm}}$ solve (22.3), and therefore they are Bernstein-gamma functions. To provide a link between the Bernstein-gamma functions associated to the Wiener-Hopf factors $\phi_\pm$ and the probabilistic $\kappa_\pm$, we use that by [PS18, Theorem 4.1 (5)], for any $c > 0$, $z \in \mathbb{C}_{(0,\infty)}$, and a Bernstein function ϕ, $W_{c\phi}(z) = c^{z-1}W_\phi(z)$, so we obtain, from the choice of $\phi_{q,\pm}$ made above, that

$$W_{\phi_{q,+}}(z) = c_+^{1-z} W_{\kappa_+}(q,z), \quad \text{and} \quad W_{\phi_{q,-}}(z) = c_-^{1-z} h^{z-1}(q) W_{\kappa_-}(q,z). \tag{22.11}$$

Substituting into (22.10) and using $c_+c_- = 1$, we get

$$\mathcal{M}_{I_{\xi,q}}(z) = h^z(q)\kappa_-(q,0)\frac{\Gamma(z)}{W_{\kappa_+}(q,z)} W_{\kappa_-}(q,1-z). \tag{22.12}$$

22.3 Methodology

Recall that

$$I_\xi(t) := \int_0^t e^{-\xi_s} \mathrm{d}s.$$

From (22.10), for $a \in (0,1)$, we have that the Laplace transform for the moments in t is given by

$$\int_0^\infty e^{-qt}\mathbb{E}\big[I_\xi^{-a}(t)\big]\mathrm{d}t = \frac{\mathcal{M}_{I_{\xi,q}}(1-a)}{q} = \frac{M_\Psi(q,1-a)}{\phi_+(q,0)}. \tag{22.13}$$

The identity (22.13) is the starting point of our proof, as it opens the way to apply Tauberian theorems to determine the asymptotic behaviour of $\mathbb{E}\big[I_\xi^{-a}(t)\big]$. However, since we work under the assumption that $\mathbb{E}[\xi_1] < 0$, it is valid that

$$\int_0^\infty \mathbb{E}\big[I_\xi^{-a}(t)\big]\mathrm{d}t < \infty,$$

see Theorem 23.6, and a direct application of Tauberian theorems is impossible. For this purpose, depending on the index of regular variation $\alpha > 1$ of $\mathbb{P}(\xi_1 > t)$, we resort to investigating several repeated tails, for example, when $\alpha \in (n, n+1]$, $n \geq 1$, we consider

$$V(t) = \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \mathbb{E}\big[I_\xi^{-a}(s_n)\big]\mathrm{d}s_n \ldots \mathrm{d}s_1,$$

and via Tauberian methods, we deduce the asymptotic behaviour of $V(t)$, as $t \to \infty$. When $\alpha \neq n+1$ by the monotone density theorem, we infer the asymptotics of $\mathbb{E}\big[I_\xi^{-a}(t)\big]$. Dealing with Laplace transforms of repeated tails requires consideration of the derivatives in q of $M_\Psi(q, 1-a)$, see (22.13), including at $q = 0$, which through (22.10) depend on the derivatives in q of the involved bivariate Bernstein-gamma functions. Via an integral representation of the bivariate Bernstein-gamma functions which links these functions to the convolutions and the derivatives of the q-potential measures of ξ, we derive results of independent interest for both the aforementioned derivatives and the potential measures.

The outstanding case is where $\alpha = n+1$ is an integer. Then the results concerning the repeated tails can be deduced as above, but the monotone density theorem fails. Indeed, $\int_0^t V(s)\mathrm{d}s$ is slowly varying at infinity, see below (26.20), and Theorem B.11 does not hold. In this case, we are forced to resort to the more complicated de Haan theory, which requires a preliminary estimate of the type $\mathbb{E}\left[I_\xi^{-a}(t)\right] = \mathrm{O}(\mathbb{P}(\xi_1 > t))$, and which we derive by purely probabilistic means such as, for example, the one-large jump principle, see Lemma 26.3. This allows us to use a generalised monotone density theorem, see Theorem B.4. The rest is similar to the non-integer α case.

Having established the asymptotics of $\mathbb{E}\left[I_\xi^{-a}(t)\right]$, we proceed to study the behaviour of

$$\frac{\mathbb{E}\left[I_\xi^{-a}(t)\mathbb{1}_{\left\{I_\xi(t)\le x\right\}}\right]}{\mathbb{P}(\xi_1 > t)}, \quad \text{for any fixed } x \in (0,\infty).$$

We do so via a Mellin inversion representation of the Laplace transform

$$\begin{aligned} x \mapsto \frac{1}{q}\mathbb{E}\left[I_\xi^{-a}(\mathbf{e}_q)\mathbb{1}_{\left\{I_\xi(\mathbf{e}_q)\le x\right\}}\right] &= \int_0^\infty e^{-qt}\mathbb{E}\left[I_\xi^{-a}(t)\mathbb{1}_{\left\{I_\xi(t)\le x\right\}}\right]\mathrm{d}t \\ &= -\frac{1}{2\pi i\phi_+(q,0)}\int_{\mathtt{Re}(z)=b}\frac{x^{-z}}{z}M_\Psi(q,z+1-a)\mathrm{d}z, \end{aligned}$$

and a subsequent investigation of nth repeated tail of $\mathbb{E}\left[I_\xi^{-a}(t)\mathbb{1}_{\left\{I_\xi(t)\le x\right\}}\right]$ as in the case above, which corresponds to the limit case $x = \infty$. Note that here the differentiation of the Laplace transform involves differentiation under the sign of the integral. This requires an additional understanding of the growth rate of the derivatives of bivariate Bernstein-gamma functions along complex contours which we develop in Theorem 24.3 and Corollary 24.4. The case when ξ is a compound Poisson process with positive drift requires further refinement of the aforementioned estimates, see item *(iii)* of Corollary 24.4. Then we are able to apply the same Tauberian, including the de Haan theory, arguments as in the case of the moments above. We emphasise that for $\alpha \in (n, n+1]$, the main carrier of the asymptotics is the behaviour of $(\phi_+(q,0))^{(n)}$ at zero. As a result, we prove that

$$\frac{\mathbb{E}\left[I_\xi^{-a}(t)\mathbb{1}_{\left\{I_\xi(t)\le x\right\}}\right]}{\mathbb{P}(\xi_1 > t)}$$

converges to the cumulative distribution function of a finite measure ν_a and compute its Mellin transform, see Theorems 23.1 and 23.5. The latter allows us to deduce analytic properties for the density of the limiting measure and to obtain a product factorisation of the random variable behind the normed ν_a, that is, $\tilde{\nu}_a$, in terms of well-known independent random variables, namely the classical exponential functional of the subordinator related to the Wiener-Hopf factor $\phi_{0,+}$ and the remainder term pertaining to $\phi_{0,-}$, see Theorem 23.5.

The bounds on the rate of decay of $\mathbb{E}[|F(I_\xi(t))|]$ in Theorem 23.6 are simple to derive from (22.13) once we have established the finiteness of the derivatives at $q = 0$ of the right-hand side of (22.13). The latter yields the finiteness of the derivatives of the integral to the left-hand side.

22.4 Background and previous results

In this section of the thesis some of the results discussed in Section 6 are examined in more detail.

23 Main results on exponential functionals

We state the main result on $I_\xi(t)$.

Theorem 23.1. *Let ξ be a Lévy process with a finite negative mean and*

$$\mathbb{P}(\xi_1 > t) \overset{\infty}{\sim} \frac{\ell(t)}{t^\alpha} \tag{23.1}$$

for $\alpha > 1$ and $\ell \in SV_\infty$. Then, for any $a \in (0,1)$,

$$\frac{t^\alpha \mathbb{P}(I_\xi(t) \in \mathrm{d}y)}{y^a \ell(t)} \xrightarrow[t\to\infty]{w} \nu_a(\mathrm{d}y), \tag{23.2}$$

where ν_a is a finite measure, supported on $(0,\infty)$, with a distribution function given by

$$\nu_a((0,x]) = \frac{1}{C}\int_{\mathtt{Re}z=b} \frac{x^{-z}}{z} M_\Psi(0, z+1-a)\mathrm{d}z, \quad \textit{with} \quad C = -2\pi i \phi_+(0,0)(-\mathbb{E}[\xi_1])^\alpha \tag{23.3}$$

and for all $b \in \mathbb{C}_{(a-1,0)}$.

A direct consequence of the last theorem are the following results.

Corollary 23.3. *Under the conditions of Theorem 23.1, for every function $F : (0,\infty) \to \mathbb{R}$ such that, for some $a \in (0,1)$, $x \mapsto x^a F(x)$ is bounded and continuous,*

$$\frac{t^\alpha}{\ell(t)}\mathbb{E}[F(I_\xi(t))] \xrightarrow[t\to\infty]{} \int_{(0,\infty)} y^a F(y) \nu_a(\mathrm{d}y) < \infty. \tag{23.4}$$

Corollary 23.4. *Under the conditions of Theorem 23.1, for any $a \in (0,1)$,*

$$\frac{t^\alpha}{\ell(t)}\mathbb{E}\left[I_\xi^{-a}(t)\right] \xrightarrow[t\to\infty]{} \frac{M_\Psi(0,1-a)}{\phi_+(0,0)(-\mathbb{E}[\xi_1])^\alpha} < \infty. \tag{23.5}$$

Theorem 23.1 and Corollary 23.4 allow the following neat probabilistic reformulation which contains an extension of the classical factorisation of exponential functionals, see [PS18, Theorem 2.22]

$$I_{\xi,q} = I_{\phi_{q,+}} \times Y_{\phi_{q,-}},$$

where $Y_{q,\phi-}$ is a positive random variable and $I_{\phi_{q,+}}$ stands for the exponential functional of the subordinator associated with $\phi_{q,+}$. Let us introduce, for $a \in \mathbb{R}$ such that $\mathbb{E}[X^a] < \infty$, the size-biased transform via, for any bounded and continuous function f,

$$\mathbb{E}[f(\mathcal{B}_a X)] = \frac{\mathbb{E}[X^a f(X)]}{\mathbb{E}[X^a]}. \tag{23.6}$$

Then we have the following factorisation.

Theorem 23.5. *Under the conditions of Theorem 23.1, for any $a \in (0,1)$,*

$$\mathcal{B}_{-a} I_\xi(t) \xrightarrow[t\to\infty]{d} \mathcal{B}_{-a} I_{\phi_+} \times \mathcal{B}_{1-a} Y_{\phi_-}^{-1}, \tag{23.7}$$

where $\times$ stands for product of independent random variables, $I_{\phi_{q,+}}$ is the classical exponential functional related to the killed subordinator with Bernstein function $\phi_+ := \phi_{0,+}$, *and Y_{ϕ_-} is the positive random variable whose Mellin transform is W_{ϕ_-} with Bernstein function $\phi_- := \phi_{0,-}$, see* (22.11)*.*

Moreover, the law of $\mathcal{B}_{-a} I_{\phi_+} \times \mathcal{B}_{1-a} Y_{\phi_-}^{-1}$ and therefore ν_a has infinitely differentiable bounded densities as long as ξ is not a compound Poisson process with positive drift.

The next result has no general assumptions on ξ apart from it being transient and despite the fact that it only offers $\mathrm{o}(\cdot)$-type of estimate, it frequently suffices, see e.g. the applications in the area of branching processes in Lévy random environments in [PPS16, BPS21, Xu21]. Following [DM04], for $x > 0$ and the linear term γ, defined in (22.4), we introduce the function

$$A(x) = \gamma + \Pi(1,\infty) - \Pi(-\infty,-1) + \int_1^x (\Pi(y,\infty) - \Pi(-\infty,-y))\mathrm{d}y. \tag{23.8}$$

From [DM04, Lemma 13 *(ii)*], in the case where $\lim_{t\to\infty} \xi_t = -\infty$,

$$\text{if} \quad \mathbb{E}[|\xi_1|] < \infty, \quad \text{then} \quad \lim_{x\to\infty} A(x) = \mathbb{E}[\xi_1] < 0;$$

and

$$\text{if} \quad \mathbb{E}[\xi_1] \text{ is not finite, then} \quad \lim_{x\to\infty} A(x) = -\infty.$$

Note that because $\lim_{x\to\infty} |A(x)| = |A(\infty)| > 0$, there exists x_0 such that $A(x)$ is non-zero for $x \geq x_0$. Since the integral criterion in (24.6), that is

$$\int_{(1,\infty)} \left(\frac{x}{|A(x)|}\right)^{n+1} \Pi(\mathrm{d}x) < \infty,$$

is in fact borrowed from [DM04, (1.14)], which requires integrability only at infinity, the possibility that it may be the case that $A(x) = 0$ for some $x \in (1, x_0]$ can without loss of generality be avoided by redefining $A(x)$ to be constant over $(1, x_0]$. From now on, we work as if $x_0 = 1$. We are ready to state our final result for this section.

Theorem 23.6. *Let ξ be a Lévy process such that $\lim_{t\to\infty} \xi_t = -\infty$. If*

$$\int_{(1,\infty)} \left(\frac{x}{|A(x)|}\right)^{n+1} \Pi(\mathrm{d}x) < \infty \tag{23.9}$$

for some $n \geq 1$, then, for every measurable function $F : (0,\infty) \to \mathbb{R}$ such that for some $a \in (0,1)$, $x \mapsto x^a F(x)$ is bounded, we have that at infinity

$$\mathbb{E}[|F(I_\xi(t))|] = \mathrm{o}\big(t^{-n}\big), \quad \text{and} \quad \int_0^\infty t^n \mathbb{E}[|F(I_\xi(t))|]\mathrm{d}t < \infty. \tag{23.10}$$

Finally, only upon the assumptions that $\lim_{t\to\infty} \xi_t = -\infty$ and $x \mapsto x^a F(x)$ is bounded, it holds that

$$\int_0^\infty \mathbb{E}[|F(I_\xi(t))|]\mathrm{d}t < \infty.$$

24 Derivatives of bivariate Bernstein–gamma functions

In this section, we study the finiteness of the derivatives of $W_{\kappa_\pm}$ and related quantities, which play a key role in the proofs in Section 26 of the thesis. However, since they are of independent interest, we state them in this separate section. We start with an integral representation of $W_{\kappa_\pm}$ recalling that, for $q \geq 0$,

$$U_q(\mathrm{d}x) = \int_0^\infty e^{-qt}\mathbb{P}(\xi_t \in \mathrm{d}x)\mathrm{d}t \tag{24.1}$$

are the q-potential measures of ξ and U_q^{*n} are their respective formal convolutions. $U := U_0$ is called the potential measure which is a well-defined Radon measure when ξ is transient, see Theorem 25.3.

The next lemma allows us to study the derivatives of $W_{\kappa_\pm}$ via a link with the potential measures of the process ξ.

Lemma 24.1. *Let ξ be a Lévy process with Lévy exponent Ψ, and $\kappa_\pm$ be the bivariate Laplace exponents associated with Ψ, introduced in* (22.5)*. Then, for any $z \in \mathbb{C}_{(-1,\infty)}$ and $q > 0$,*

$$\log\big(W_{\kappa_\pm}(q, z+1)\big) = z\ln(\kappa_\pm(q,1)) + \int_{[0,\infty)} \big(e^{-zy} - 1 - z(e^{-y}-1)\big)\frac{\mathcal{W}_\pm(q,\mathrm{d}y)}{e^y - 1}, \tag{24.2}$$

where log *is a complex logarithm, and the measures $\mathcal{W}_\pm(q,\mathrm{d}y)$ do not charge $\{0\}$ and are defined on $\mathbb{R}\setminus\{0\}$ as*

$$\mathcal{W}_\pm(q,\mathrm{d}y) = \int_0^\infty \frac{e^{-qt}}{t}\mathbb{P}(\xi_t \in \pm\mathrm{d}y)\mathrm{d}t. \tag{24.3}$$

Also, for any $z \in \mathbb{C}_{(-1,\infty)}$ and $q > 0$, we have the representation

$$\frac{\frac{\partial}{\partial q}W_{\kappa_\pm}(q,1+z)}{W_{\kappa_\pm}(q,1+z)} = \int_{[0,\infty)} \frac{1-e^{-zy}}{e^y - 1}U_q(\pm\mathrm{d}y). \tag{24.4}$$

Finally, for any $q_0 > 0$, all derivatives of $\mathcal{W}_\pm(q,\mathrm{d}y)$ in q exist at q_0 and are equal to

$$\begin{aligned}\mathcal{W}_\pm^{(n)}(q_0,\mathrm{d}y) &= (-1)^n\int_0^\infty t^{n-1}e^{-q_0t}\mathbb{P}(\xi_t\in\pm\mathrm{d}y)\mathrm{d}t\\ &= (-1)^n(n-1)!U_{q_0}^{*n}(\pm\mathrm{d}y)\\ &= -U_{q_0}^{(n-1)}(\pm\mathrm{d}y),\end{aligned} \tag{24.5}$$

where $(\partial^{n-1}/\partial q^{n-1})U_{q_0} = U_{q_0}^{(n-1)}$ are the measure derivatives of U_q at $q = q_0$.

Theorem 24.3. *Let ξ be a Lévy process with Lévy-Khintchine exponent Ψ such that $\lim_{t\to\infty}\xi_t = -\infty$. Then, for any $n \geq 0$, $z \in \mathbb{C}_{(-1,\infty)}$, and $q \geq 0$,*

$$\int_{(1,\infty)}\left(\frac{x}{|A(x)|}\right)^{n+1}\Pi(\mathrm{d}x) < \infty \iff \frac{\partial^{n+1}}{\partial q^{n+1}}W_{\kappa_\pm}(q,z+1) \text{ and } \frac{\partial^{n+1}}{\partial q^{n+1}}W_{\phi_{q,\pm}}(z+1) \text{ are finite,} \tag{24.6}$$

where for $q = 0$ the derivatives are understood as right derivatives. Upon the finiteness of the integral condition in (24.6)*, the derivatives are right-continuous at $q = 0$.*

Next, we provide estimates for the q-derivatives of the quantity, defined in (22.10),

$$M_\Psi(q,z) := \frac{\Gamma(z)}{W_{\phi_{q,+}}(z)} W_{\phi_{q,-}}(1-z),$$

which is vital for understanding the Mellin transform of $I_{\xi,q}$.

Corollary 24.4. *Let ξ be a Lévy process with Lévy-Khintchine exponent Ψ such that $\lim_{t\to\infty}\xi_t = -\infty$,*

$$M_\Psi(q,z) := \frac{\Gamma(z)}{W_{\phi_{q,+}}(z)} W_{\phi_{q,-}}(1-z),$$

and assume that, for some $n \geq 0$,

$$\int_{(1,\infty)} \left(\frac{x}{|A(x)|}\right)^{n+1} \Pi(\mathrm{d}x) < \infty. \tag{24.7}$$

Then, for any $0 \leq k \leq n+1$, $z \in \mathbb{C}_{(0,1)}$, and $q \geq 0$:

(i) the derivatives $\frac{\partial^k}{\partial q^k} M_\Psi(q,z)$ are finite and right-continuous at zero;

(ii) there exist polynomials $P_{\mathtt{Re}(z),k}$ of degree k such that

$$\left|\frac{\partial^k}{\partial q^k} M_\Psi(q,z)\right| \leq P_{\mathtt{Re}(z),k}(|z|)|M_\Psi(q,z)|;$$

(iii) if the potential measure of ξ has a bounded density with respect to the Lebesgue measure, there exist polynomials $P_{\mathtt{Re}(z),k}$ of degree k such that

$$\left|\frac{\partial^k}{\partial q^k} M_\Psi(q,z)\right| \leq P_{\mathtt{Re}(z),k}(|\ln|z||)|M_\Psi(q,z)|.$$

Moreover, if in addition $\mathbb{E}[\xi_1] \in (-\infty,0)$, $n \geq 1$, and for any $0 < \beta < n$,

$$\int_{(1,\infty)} x^{\beta+1}\Pi(\mathrm{d}x) < \infty = \int_{(1,\infty)} x^{n+1}\Pi(\mathrm{d}x), \tag{24.8}$$

then

(i') items (i)-(iii) hold for any $0 \leq k \leq n$ and $q \geq 0$. For $k > n$, the derivatives $\frac{\partial^k}{\partial q^k} M_\Psi(q,z)$ exist for $q > 0$.

For $k = n+1$, $q > 0$, and $z \in \mathbb{C}_{(0,1)}$, items (ii)-(iii) are modified to

(ii') for any $\delta > 0$, there exist polynomials $P_{\mathtt{Re}(z),n,\delta}$ of degree $n+1$ such that, for all $q > 0$,

$$\left|\frac{\partial^{n+1}}{\partial q^{n+1}} M_\Psi(q,z)\right| \leq \left(U_q^{*n}(\mathbb{R}_+) + q^{-\delta}\right) P_{\mathtt{Re}(z),n,\delta}(|z|)|M_\Psi(q,z)|;$$

(iii') if the potential measure of ξ has a bounded density with respect to the Lebesgue measure, for any $\delta > 0$, there exist polynomials $P_{\mathtt{Re}(z),n,\delta}$ of degree $n+1$ such that, for all $q > 0$,

$$\left|\frac{\partial^{n+1}}{\partial q^{n+1}} M_\Psi(q,z)\right| \leq \left(U_q^{*n}(\mathbb{R}_+) + q^{-\delta}\right) P_{\mathtt{Re}(z),n,\delta}(|\ln|z||)|M_\Psi(q,z)|;$$

(iv') it holds true that $\lim_{q\to 0} U_q^{*n}(\mathbb{R}_+) = \infty$.

The last result shows that, under the condition of Corollary 24.4 and in the setting of item *(iii)*, we have a general estimate for the densities of the convolutions of the q-potential measures.

Theorem 24.6. *Let ξ be a Lévy process with Lévy-Khintchine exponent Ψ such that $\lim_{t\to\infty} \xi_t = -\infty$. Also, assume that U has a bounded density u on $\mathbb{R}$ with $C := \sup_{x\in\mathbb{R}} u(x)$. Then, for any $k \geq 1$ and $q > 0$, the potential densities of U_q^{*k} exist and are locally bounded, since they satisfy, for any $x \in \mathbb{R}$ and $q > 0$,*

$$u_q^{*k}(x) \leq kCU_q^{*(k-1)}(\min\{0, x\}, \infty) \tag{24.12}$$

with the convention $U^{*0} \equiv 1$. Next, if for some $n \geq 1$,

$$\int_{(1,\infty)} \left(\frac{x}{|A(x)|}\right)^{n+1} \Pi(\mathrm{d}x) < \infty,$$

then, for any $1 \leq k \leq n+1$, we have that the potential densities of U^{*k} exist. Moreover, for every $x \in \mathbb{R}$, $U^{*(k-1)}(x, \infty) < \infty$ and (24.12) holds with $q = 0$.

25 Proofs on Bernstein-gamma functions

We proceed with the proofs for Section 24 by stating a preliminary proposition, which links the derivatives of the q-potentials, see (24.1), with their convolutions.

Proposition 25.1. *Let U_q be the potential measures of a Lévy process. Then, for every $q \geq 0$ and $n \geq 1$,*

$$U_q^{*n}(\mathrm{d}x) = \frac{1}{(n-1)!}(-1)^{n-1}U_q^{(n-1)}(\mathrm{d}x) = \frac{1}{(n-1)!}\int_0^\infty e^{-qt}t^{n-1}\mathbb{P}(\xi_t \in \mathrm{d}x)\mathrm{d}t, \tag{25.1}$$

with the measures necessarily finite for $q > 0$ and possibly infinite for some or all n when $q = 0$.

We prove Proposition 25.1, as well as the next technical lemma, in Section 27.

Lemma 25.2. *For $z \in \mathbb{C}_{(-1,\infty)}$, the functions*

$$u_z(y) := \frac{e^{-zy} - 1 - z(e^{-y} - 1)}{e^y - 1}, \quad \text{and} \quad v_z(y) := ze^{-y} - u_z(y) = \frac{1 - e^{-zy}}{e^y - 1} \tag{25.2}$$

are bounded and continuous on $[0, \infty)$. Furthermore, there exist $C_{\mathtt{Re}(z)}$, $C_{1,\mathtt{Re}(z)}$, $C_{2,\mathtt{Re}(z)}$, $\epsilon_{1,\mathtt{Re}(z)}$, $\epsilon_{2,\mathtt{Re}(z)} > 0$ such that, for all $y \geq 0$ and $x \geq C_{\mathtt{Re}(z)}$,

$$|v_z(y)| \leq C_{1,\mathtt{Re}(z)}|z|e^{-\epsilon_{1,\mathtt{Re}(z)}y}, \quad \text{and} \quad |v_z(x)| \leq C_{2,\mathtt{Re}(z)}e^{-\epsilon_{2,\mathtt{Re}(z)}x}. \tag{25.3}$$

In the next theorem, we also outline some basic facts about potential measures, which can be found in [Rev84]. Its first part is [Rev84, p. 101, Corrolary 3.3.6], and the second one is a combination of [Rev84, p. 169, Theorem 5.3.1], [Rev84, p. 171, Theorem 5.3.4], and [Rev84, p. 173, Theorem 5.3.8].

Theorem 25.3. *Let ξ be a Lévy process which drifts to $-\infty$.*

1. *The measures U are Radon measures, i.e., for every compact $K \subset \mathbb{R}$, $U(K) < \infty$.*
2. *(renewal theorem) If ξ is non-lattice, then it holds that vaguely $\lim_{x\to\infty} U(x-\mathrm{d}y) = 0\mathrm{d}y$, and $\lim_{x\to-\infty} U(x-\mathrm{d}y) = -\mathrm{d}y/\mathbb{E}[\xi_1]$ with the convention $1/\infty = 0$. In the case of a lattice process, $\mathrm{d}y$ should be replaced by the counting measure on the respective lattice.*

The thesis continues with the proof of Lemma 24.1, Theorem 24.3, Corollary 24.4, and Theorem 24.6.

26 Proofs on exponential functionals

The following three auxiliary results are used in the proof of Theorem 23.1.

Proposition 26.1. *Let $f : [0,\infty) \to [0,\infty)$, and define $f_0 := f$. For $x \geq 0$ and $n \geq 1$, define*

$$f_n(x) := \int_x^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty f(s_n)\mathrm{d}s_n \ldots \mathrm{d}s_1.$$

Then, for $q \geq 0$,

$$\widehat{f_n}(q) = \frac{(-1)^n}{(n-1)!} \int_0^1 \widehat{f}^{(n)}(qv)(1-v)^{n-1}\mathrm{d}v \tag{26.1}$$

in the extended with $\infty = \infty$ sense.

Proposition 26.2. *Let f be a regularly varying function at zero with index $\beta \in (-1,0]$ and $n \geq 1$. Then*

$$\int_0^1 f(qv)(1-v)^{n-1}\mathrm{d}v \overset{0}{\sim} f(q)\int_0^1 v^\beta(1-v)^{n-1}\mathrm{d}v = f(q)\frac{\Gamma(n)\Gamma(\beta+1)}{\Gamma(n+\beta+1)}. \tag{26.2}$$

Lemma 26.3. *Let ξ be a Lévy process with a finite negative mean and $\mathbb{P}(\xi_1 > t) \overset{\infty}{\sim} \ell(t)/t^\alpha$ for some $\alpha > 1$. Then*

$$\mathbb{E}\big[I_\xi^{-a}(t)\big] = \mathrm{O}(\mathbb{P}(\xi_1 > t)) = \mathrm{O}\bigg(\frac{\ell(t)}{t^\alpha}\bigg). \tag{26.3}$$

Next, the proofs of Theorem 23.1, Theorem 23.5, and Theorem 23.6 are provided in the thesis.

27 Proofs of auxiliary results

This section contains the proofs of the auxiliary results Proposition 25.1, Lemma 25.2, Proposition 26.1, Proposition 26.2, and Lemma 26.3.

Chapter VI

Conclusion

28 Main contributions

The main contributions of this thesis, based on opinion, are the following.

1. The results on the density and tail of the exponential functional of subordinators from Section . The respective asymptotics at infinity are obtained under the mild condition of positive increase in Section 17.
2. The obtained weak limit of the law of a suitably scaled exponential functional of a Lévy process with finite negative mean and regularly varying tail, see Theorem 23.1. As a consequence, various asymptotic results on the expectations of a functional transformation of these random variables, see Section 23.
3. The improved understanding on the Stirling asymptotics of Bernstein-gamma functions, presented in Section 19 and used throughout Chapter IV.
4. The analysis of the derivatives of bivariate Bernstein-gamma functions using analytic and probabilistic arguments through a link with the potential measure and its successive convolutions of the associated Lévy process, see Section 24.

29 Publications related to the thesis

1. M. Minchev and M. Savov. *Asymptotics for densities of exponential functionals of subordinators.* Bernoulli, 29(4):3307–3333, 2023. See [MS23].

30 Approbation of the thesis

The results of the thesis have been presented as:

1. *Bivariate Bernstein-gamma functions and asymptotic behaviour of exponential functionals on deterministic horizon*, Stochastic Processes and their Applications, Lisbon, Portugal, 24-28 July 2023. Joint work with Mladen Savov. Poster based on Chapter V.

2. *Bivariate Bernstein-gamma functions and asymptotic behaviour of exponential functionals on deterministic horizon*, Mathematics Days in Sofia, 10-14 July, 2023, Sofia, Bulgaria. Joint work with Mladen Savov. Talk based on Chapter V.

3. *Bivariate Bernstein-gamma functions and asymptotic behaviour of exponential functionals on deterministic horizon*, Seminar on Advances in Statistics, 9-12 March, 2023, Veliko Tarnovo, Bulgaria. Joint work with Mladen Savov. Talk based on Chapter V.

4. *Asymptotics of densities of exponential functionals of subordinators*, Lévy Processes and Random Walks (in celebration of Ron Doney's 80th birthday), 26-28 July, 2022, Manchester, UK. Poster based on [MS23].

5. *Asymptotics of densities of exponential functionals of subordinators*, Faculty of Informatics and Mathematics' Spring Science Session, 26 March 2022, Sofia, Bulgaria. Talk based on [MS23].

The paper [MS23] have been cited in the following publications:

1. J. Arista and V. Rivero. Implicit renewal theory for exponential functionals of Levy processes. *Stochastic Process. Appl.*, 163:262–287, 2023. See [AR23].

2. J. Bertoin. On the local times of noise reinforced Bessel processes. *Ann. H. Lebesgue*, 5:1277–1294, 2022. See [Ber22].

3. B. Haas. Tail asymptotics for extinction times of self-similar fragmentations. *Ann. Inst. Henri Poincare Probab. Stat.*, 59(3):1722–1743, 2023. See [Haa23].

4. B. Haas. Precise asymptotics for the density and the upper tail of exponential functionals of subordinators. In *Séminaire de probabilites*, Lecture Notes in Math. Springer, 2024+. See [Haa24]

5. P. Patie and R. Sarkar. Weak similarity orbit of (log)-self-similar Markov semigroups on the Euclidean space. *Proc. Lond. Math. Soc.* (3), 126(5):1522–1584, 2023. See [PS23].

6. R. Sarkar. Continuous and Discrete Self-Similarity via Classification Schemes of Markov Processes, and the van Dantzig Problem. *PhD thesis, Cornell University*, 2022. See [Sar22].

7. T. Simon. A note on the α-sun distribution. *Electron. Commun. Probab.*, 28:Paper No. 19, 13, 2023. See [Sim23].

31 Declaration of originality

The author declares that this thesis contains original results obtained by him or in cooperation with his co-authors. Use of the results of other scientists is accompanied by suitable citations.

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